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THEORIES OF NEUTRINO

and the

DOUBLE BETA DECAY

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INTRODUCTION

The existence of the neutrino, postulated by Pauli in order to explain the conservation of energy, momentum and angular momentum in the process of β -decay, is today fairly well established.

A theoretical treatment of the interaction of this particle with electrons and nucleons in the process of β -decay of atomic nuclei was first given by Fermi[†, 37]. In the original paper Fermi used for the description of the neutrino a Dirac field, i.e., a field describing two types of particles: neutrinos and antineutrinos. He used an especial type of interaction which is called today a vector interaction. Four other types of interaction (scalar, pseudoscalar, etc.) were proposed by Bethe and Bacher[43]. Also Wigner and Critchfield[40] suggested an additional interaction (the antisymmetrical interaction) which can be expressed as a linear combination of the scalar, pseudoscalar and pseudovector ones. An analysis of a general type of interaction given by a linear combination of the five types of simple ones was made by Fierz[39]. In these papers, as in most of the more recent ones, the modification of the Fermi interaction proposed by Konopinski and Uhlenbeck[36], (the one not involving derivatives of the neutrino field), is preferred to Fermi's original form.

Majorana[26] has shown that the neutrino theory can be treated in a form which is in close analogy to treatment of neutral Boson fields. In the case of Boson fields non-hermitian operators are used to describe charged

†). The numbering of the references in this Introduction correspond to the order in which they appear in the text. A list of references is given at the end of the paper.

particles and two types of particles are associated to each field (positively and negatively charged); on the other hand, hermitian operators are used for neutral fields and, as a consequence, there is only one type of neutral particle. In Majorana theory the neutrino field operators are also assumed to be hermitian if an special representation of the Dirac equation is used and thus only one type of neutrino exists. This condition of self charge conjugation was extended by Furry[27] to the case of an arbitrary representation of the Dirac equation.

The main question arising from Majorana's work is the following:

Is the neutrino a Dirac particle (i.e., do both neutrinos and anti-neutrinos exist?), or is it a Majorana particle (i.e., does only one type of neutrino exist?) ? This question cannot be answered by the analysis of the single β-decay as both types of theories lead to the same results in this case.

Furry[44] was the first to point out a possibility of deciding experimentally between these two possibilities. This decision is made possible by the fact that if a given nucleus decays with the successive emission of two electrons then Majorana's theory would lead to a half life of the order of 10^{16} years, in contrast to the value of 10^{22} years that had been found by M. Goeppert Mayer[43] for the half life of the double β-decay in an usual type of theory. The reason for this difference in the values of the half life is that in Majorana theory a neutrino can be either emitted or absorbed in a given elementary process. For instance, we have both possibilities:

$$N \longrightarrow P + e + \nu \qquad (1)$$

$$N + \nu \longrightarrow P + e \qquad (2)$$

where N, P, e and ν represent, respectively, the neutron, proton (+), electron (or, better, negaton) and neutrino,(1) and (2) are to be contrasted with the corresponding processes for the usual type of Fermi interaction (used by M. G. Mayer) in which the neutrino is a Dirac particle.

$$N \longrightarrow P + e + \nu \qquad (1)$$

$$N + \nu' \longrightarrow P + e \quad \text{(but not (2)!)} \qquad (2a)$$

where ν' represents an antineutrino (in this last case we could also interchange ν and ν').

Thus in double β decay, if the neutrino is a Majorana particle, a neutrino can be emitted in the first decay (virtual transition) and reabsorbed in the second one. This increases considerably the number of intermediate states, as this neutrino can have any energy in opposition to the case computed by M. G. Mayer in which the energy of the emitted neutrinos are restricted by the conservation of energy.

The brilliant experiment performed by Fireman[51] made the decision in favour of Majorana's theory and opened a new chapter - that of the double β-decay - in Nuclear Physics.

However, this experiment does not prove decisively that the neutrino is not a Dirac particle. For Toushek[38] has shown that even if the neutrino is a Dirac particle, we can obtain double β-decay with no neutrino emission, provided that we include in the interaction two convenient types of terms, one in which a neutrino (antineutrino) is emitted (absorbed) together with an electron and one in which an antineutrino (neutrino) is emitted (absorbed) with an electron. Thus in this case both (1) and (2) holds again (as in Majorana theory). Also the processes obtained from (1) and (2) by substitution ν' for ν are possible. A theory of this type had been already suggested by Racah[20]. Fireman[9] also considered one such theory. He obtained, however, a positive result, in disagreement with Toushek who found that no double β decay without neutrino emission can occur in the type of theory considered by Fireman. The reason for this discrepancy is to be traced to the fact that Fireman omitted the contributions from the antineutrino to the process, which in this particular theory cancelled the contributions arising from the neutrinos.

The previously existing analyses of theories with two types of neutrino leading to double β decay without emission of neutrinos are not, however, completely satisfactory, because only some special cases have been examined. For instance, the simple type of theory suggested by Racah[20] has not been considered. Also all of these computations either contain some mistakes or inappropriate approximations. For instance, in Furry's computation (Majorana theory) not only are some terms neglected but also the unsatisfactory assumption is made that only one state of the intermediate nucleus yields a large contribution to the transition probability. Toushek, on the other hand, fails to antisymmetrize the two-electron wave function for the final state. All of them neglect the selection rules which play an important role in this process, since both the initial and final nuclei are of the even-even type.

The purpose of this paper is as follows:

1) To try to formulate a theory of the type considered by Fireman in which, however, only neutrinos (and no antineutrinos) would be involved in the interaction. Such a theory would lead in a correct way to the same results that Fireman mistakenly obtained in his computations. However, we will discard this possibility (Part I) as the resulting theory will be shown not to be relativistic invariant.

2) To formulate possible theories of neutrinos, with special view to those leading to double β decay with no emission of neutrinos (Part IV). No derivatives of the neutrino field will be included in the β-decay interaction as in the theory proposed by Konopinski and Uhlenbeck[36] in view of the fact that such type of theories are in disagreement with the experimental results on double β-decay. An analysis of the relativistic invariance of the field theories (Part II) and of the underlying Hilbert space of wave functions (Part III) will be also made.

3) To compute the probability for double β-decay with no neutrinos in the several types of theory and to analyze the experimental results (Part V).

PART I

FIELD THEORIES

The formulation of the quantum field theories, sometimes referred to as "second quantization", has received recently significant contributions, especially from the works of Tomonaga(1) and Schwinger(2), who introduced the "interaction representation", and from those of Feynman(3) and Stueckelbe g(4), who succeeded in finding the S-matrix formulation of the Quantum Electrodynamics in configuration space in the lines suggested by Heisenberg(5). The equivalence of these formulations was shown by Dyson(6). These works lead not only to a better understanding of the field theories but, at least in the case of Quantum Electrodynamics to computational prescriptions leading to finite answers for problems whose solution in the earlier stages of the theory was impaired by the presence of infinite terms.

All these treatments, as well as the earlier works, were in the frame of local field theories whose characteristics were made precise recently by Dirac(7). An attempt to consider non-localizable fields was made recently by Yukawa(8). Non-localized interactions has also been tried by several authors(9) in order to eliminate the divergencies without subtraction procedures.

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(1). S. Tomonaga, Prog. Theor Phys., 1, 27, 1946.

(2). J. Schwinger, Phys. Rev. 74, 1439, 1948; 75, 651, 1949; 76, 790, 1949.

(3) R.P. Feynman, Phys. Rev. 76, 749, 769, 1949.

(4) Compare E.C.G. Stueckelberg and D. Rivier, Helv.Phys. Acta, 23,215,1949.

(5) W.Heisenberg, Zeits.f. Phys. 120, 513,673,1943; Zeits.f. Naturf., 1, 608,1946. See also J.A. Wheeler, Phys. Rev. 52,1107,1937.

(6) F.J. Dyson, Phys. Rev. 75, 486,1736,1949.

(7) P.A.M. Dirac, Phys. Rev. 73, 1092, 1948.

(8) H. Yukawa, Phys. Rev. 77, 219, 849,1950.

(9) A. Pais and G. Uhlenbeck, Phys. Rev. 79, 145, 1950 (with an extensive bibliography.)

We shall restrict ourselves to local field theories (a review of which will be made in section A of this Part), although an attempt will be made in section B, with negative result, to consider a special type of non-local theory of the neutrino. This case was suggested by Fireman's computation of double β -decay[10] in a special theory which will be analysed later. As will be seen Fireman's theory does not lead to double β decay (for a zero mass of the neutrino) in view of the cancellation between corresponding terms in which the intermediate neutrino is either a particle or an anti-particle. However, in Fireman's computation the second type of terms were suppressed, thus leading to a finite result. This would correspond in field theory to suppressing the negative energy part of the neutrino field operator, thus leading to a non-local theory (as the positive energy part of the field quantities do not anticommute in points connected by space like vectors). This resulting theory cannot be formulated in the Interaction Representation. We did not succeed in formulating it in Heisenberg Representation either. It can be formulated in Schrodinger Representation but it is then difficult to analyse its Lorentz Invariance. We shall analyse its formulation in the S-matrix Representation and conclude against its Lorentz Invariance (section B).

A. Local field theories.

There are four equivalent formulation of such theories: Heisenberg Representation, Interaction Representation, Schrodinger Representation and S-Matrix formulation. We shall start with the Interaction Representation

(10) E. L. Fireman, "An experiment on double β -decay," Princeton Thesis, 1948.

not only because the commutation relations are quite simple in this representation, but also because it is the most appropriate form to the analysis of the Lorentz Invariance.

1). Interaction Representation.

If we have several tensor (scalar, vector, tensor, etc.) fields $A^{(k)}$ (if hermitian they will correspond to neutral quanta, if not to charged ones) and several spinor fields $\psi^{(\ell)}_{(x)}$, which are functions only of the space-time coordinates: $x = (x^\mu) = (x^0 = t, x_1, x_2, x_3)$ the field equations will be in the interaction representation:

$$(\gamma^\mu \frac{\partial}{\partial x^\mu} + m^{(\ell)})\psi^{(\ell)}(x) = 0 \qquad \text{(I-1)}$$

$$(\Box - m^2_{(k)})\, A^{(k)}(x) = 0 \qquad \text{(I-2)}$$

$$i\,\frac{\delta \Psi[\sigma]}{\delta\sigma(x)} = \mathcal{H}(x)\,\Psi[\sigma] \qquad \text{(I-3)}$$

where $\Psi[\sigma]$ is the quantum wave function of the system, at the space-like surface σ, and:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,g^{\mu\nu}, \qquad \text{(I-4)}$$

$$\Box = g^{\mu\nu}\frac{\partial}{\partial x^\nu}\,\frac{\partial}{\partial x^\mu}\,; \qquad \text{(I-5)}$$

$$g^{\mu\nu} = 0 \text{ if } \mu \neq \nu\,;\ = 1 \text{ if } \mu = \nu = 1,2,3;\ = -1 \text{ if } \mu = \nu = 0 \quad \text{(I-6)}$$

Here $\hbar = c = 1$; repeated upper and lower indice mean summation from 0 to 3. $\mathcal{H}(x)$ is an scalar hermitian operator formed with the field quantities. Together with these equations the commutation relations are to be given:

$$\left[A^{(k)}(x)\,,\, A^{(k')}(x')^\dagger\right] = A^{(k)}(x)A^{(k')}(x')^\dagger - A^{(k')}(x')^\dagger A^{(k)}(x) = \delta_{kk'}\,G(x-x'), \qquad \text{(I-7)}$$

where the symbol $\dagger$ means "hermitian conjugate".

$G(x-x')$ is a tensor of the same rank as $A(x)A(x')$; if $A^{(k)}$ is a scalar field then $G(x-x') = \frac{1}{i}\Delta(x-x')$ (we use so far as possible Schwinger's notation[2]).

$$\left\{\psi_{\alpha}^{(\ell)}(x), \overline{\psi_{\beta}^{(\ell')}}(x')\right\} = \psi_{\alpha}^{(\ell)}(x)\overline{\psi_{\beta}^{(\ell')}}(x') + \overline{\psi_{\beta}^{(\ell')}}(x')\psi_{\alpha}^{(\ell)}(x) = \frac{1}{i}\,\delta_{\ell\ell'}\,S_{\alpha\beta}(x-x') \qquad (I-8)$$

where: $S(x-x') = \left(\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} - m\right)\Delta(x-x')$ (I-9)

and: $\overline{\psi} = -i\,\psi^{\dagger}\gamma^{0} = \psi^{\dagger}\beta = \beta^{T}\psi^{\dagger}$ (I-10)

For the moment we shall exclude the case when some of the spinor fields do satisfy a condition of the type: $\psi = \pm C\,\overline{\psi}$ (self charge conjugation), where:

$$C\,\overline{\psi}(x) = \psi'(x)$$

is the charge conjugate field, the matrix C being defined by

$$(C\gamma^{\mu}C^{-1})_{\alpha\beta} = -\gamma^{\mu}_{\beta\alpha}$$

In other words we restrict ourselves, for the moment, to Dirac type of spinor fields (with particle and antiparticle).

The other commutators (or anticommutators, if between spinor fields) are equal to zero.

In this representation the momentum-energy vector is:

$$P_{\nu}[\sigma] = P_{\nu}^{(o)} - \int_{\sigma} d\sigma_{\mu}\,\mathcal{H}(x), \qquad (I-11)$$

σ being an spacelike surface and:

$$d\sigma_{\mu} = (dx^{1}dx^{2}dx^{3},\ dx^{2}dx^{3}dx^{0},\ dx^{3}dx^{0}dx^{1},\ dx^{0}dx^{1}dx^{2}). \qquad (I-12)$$

$P_{\nu}^{(o)}$ is the energy-momentum vector for free fields:

$$P_{\nu}^{(o)} = \int d\sigma^{\mu}\,T_{\mu\nu}^{(o)}(x) \qquad (I-13)$$

which satisfies the conservation equation[2]:

$$\frac{\delta P_\mu^{(o)}}{\delta\sigma(x)} = \frac{\partial T_{\mu\nu}^{(o)}(x)}{\partial x_\nu} = 0 \qquad \text{(I-14)}$$

$P_\nu^{(o)}$ should satisfy the commutation relations:

$$\left[P_\mu^{(o)}, A^{(k)}(x)\right] = i\frac{\partial A^{(k)}}{\partial x^\mu} \qquad \text{(I-15)}$$

$$\left[P_\mu^{(o)}, \psi^{(\ell)}(x)\right] = i\frac{\partial \psi^{(\ell)}}{\partial x^\mu} \qquad \text{(I-16)}$$

We can also write:

$$P_\mu[\sigma] = \int_\sigma T_{\mu\nu}(x)\, d\sigma^\nu \qquad \text{(I-17)}$$

where:

$$T_{\mu\nu}(x) = T^{o}_{\mu\nu}(x) - g_{\mu\nu}\mathcal{H}(x) \qquad \text{(I-18)}$$

In this representation, if $F_\mu[\sigma]$ is an operator of the type:

$$F_\mu[\sigma] = \int_\sigma F(x)\, d\sigma_\mu \qquad \text{(I-19)}$$

it is convenient to define the derivative(2):

$$i\frac{\Delta F_\mu[\sigma]}{\Delta\sigma(x)} = i\frac{dF(x)}{dx^\mu} = i\frac{\partial F(x)}{\partial x^\mu} - \left[\mathcal{H}(x), \int_\sigma F(x')\, d\sigma'_\mu\right] \qquad \text{(I-20)}$$

The conservation law for a quantity $F_\mu(x)$ reads then:

$$i\frac{dF_\mu(x)}{dx_\mu} = i\frac{\partial F_\mu(x)}{\partial x_\mu} - \int_\sigma d\sigma'^\mu\left[\mathcal{H}(x), F_\mu(x')\right] = 0 \qquad \text{(I-21)}$$

In the special case when the condition:

$$\left[F(x), \mathcal{H}(x')\right] = 0 \quad \text{if } x-x' \text{ is spacelike,} \qquad \text{(I-22)}$$

is satisfied we have:

$$i\frac{dF(x)}{dx^\mu} = i\frac{\partial F(x)}{\partial x^\mu} - \left[\int_\sigma d\sigma'_\mu \mathcal{H}(x'), F(x)\right] = \left[P_\mu[\sigma], F(x)\right] \qquad \text{(I-23)}$$

Thus we find:

$$i\,\frac{\Delta P_\mu[\sigma]}{\Delta\sigma(x)} = i\,\frac{d\,T_{\mu\nu}(x)}{dx_\nu} = i\,\frac{\delta P_\mu[\sigma]}{\delta\sigma(x)} = \left[\mathcal{H}(x), P_\mu[\sigma]\right] = \int d\sigma'_\mu \left[\mathcal{H}(x), \mathcal{H}(x')\right]. \qquad \text{(I-24)}$$

The conservation law for energy-momentum imposes then that this expression be zero, which is a restriction on the commutation relations of the field operators.

Of course expression (I-24) will vanish if:

$$\left[\mathcal{H}(x), \mathcal{H}(x')\right] = 0 \quad \text{for } x-x' \text{ spacelike.} \qquad \text{(I-25)}$$

Thus relation (I-25) is a sufficient condition for the conservation of energy-momentum in a field theory. Clearly enough condition (I-25) is automatically satisfied in the usual theories in which $\mathcal{H}(x)$ is formed with the total field operators, whose commutators (or anticommutators) vanish for x-x' spacelike. However, this will not be true, in general, if, for some fields, only the positive energy part enters in the definition of $\mathcal{H}(x)$ or if commutators other than the usual ones are used.

Integrability condition:

In order that equation (I-3) be integrable it is necessary that the variational derivatives on two different points of an spacelike surface commute (we restrict our hamiltonian densities to those which are independent of the normals), or:

$$\left[\frac{\delta}{\delta\sigma(x')}\,\frac{\delta}{\delta\sigma(x'')} - \frac{\delta}{\delta\sigma(x'')}\,\frac{\delta}{\delta\sigma(x')}\right]\Psi[\sigma] = -\left[\mathcal{H}(x''), \mathcal{H}(x')\right]\Psi[\sigma] = 0 \qquad \text{(I-26}$$

which is true if (I-25) is satisfied, but us less restrictive than that condition. However, it seems that it would be very complicated to work out a theory with a supplementary condition like (I-26). One should observe that (I-25) imposes not only a restriction on the form of $\mathcal{H}(x)$ but also on the commutation relations of the field operators. Also it assures the conservation law for energy-momentum; nevertheless this law could read:

$$\frac{\partial T_{\mu\nu}(x)}{\partial x_\nu} \Psi[\sigma] = 0$$

- this would be the case if we had (I-26) satisfied instead of (I-25).

2) Heisenberg representation:

This is the representation in which the wave function Ψ_H is independent of σ:

$$\frac{\delta}{\delta\sigma(x)} \Psi_H = 0. \qquad \text{(I-27)}$$

Now this new wave function should be related to the old one by an unitary transformation:

$$\Psi[\sigma] = U[\sigma]\, \Psi_H \qquad \text{(I-28)}$$

such that:

$$i\, \frac{\delta U[\sigma]}{\delta\sigma(x)} = \mathcal{H}(x)\, U[\sigma] \qquad \text{(I-29)}$$

Any operator $F(x)$ of the interaction representation will go into:

$$\underset{\sim}{F}(x,\sigma) = U^{-1}[\sigma]\, F(x)\, U[\sigma], \text{ where } x \in \sigma. \qquad \text{(I-30).}$$

Now:

$$\frac{\delta}{\delta\sigma(x')} \underset{\sim}{F}(x,\sigma) = U^{-1}[\sigma]\, F(x) \frac{\delta U[\sigma]}{\delta\sigma(x')} - U^{-1}[\sigma] \frac{\delta U[\sigma]}{\delta\sigma(x')} U^{-1}[\sigma] F(x) U[\sigma]$$

$$= U^{-1}[\sigma] \left[\frac{\delta U[\sigma]}{\delta\sigma(x')}, \underset{\sim}{F}(x,\sigma) \right] \qquad \text{(I-31)}$$

Then by (I-29):

$$\frac{\delta}{\delta\sigma(x')}\,\underset{\sim}{F}(x,\sigma) = \left[\underset{\sim}{\mathcal{H}}(x',\sigma),\,\underset{\sim}{F}(x,\sigma)\right] = U^{-1}[\sigma]\left[\mathcal{H}(x'),F(x)\right]\,U[\sigma]$$

Now the usual form of the Heisenberg representation implies that also the field operators do not depend on the surface σ (although they still depend on the time coordinate).

Thus the condition that the new operator $\underset{\sim}{F}$ will not depend on σ is that:

$$\left[\mathcal{H}(x'),\,F(x)\right] = 0 \quad \text{if } x-x' \text{ is spacelike.} \qquad \text{(I-32)}$$

If this condition is satisfied for $A^{(k)}(x)$ and $\psi^{(\ell)}(x)$, or:

$$\left[\mathcal{H}(x),\,A^{(k)}(x')\right] = \left[\mathcal{H}(x),\psi^{(\ell)}(x')\right] = 0, \text{ for } x-x' \text{ spacelike,} \qquad \text{(I-33)}$$

then the equations of motion of the field quantities, in Heisenberg representation, become:

$$(\Box - m_k^2)\,\underset{\sim}{A}^{(k)}(x) = i\,U^{-1}[\sigma]\int_\sigma\left[\mathcal{H}(x),\,\frac{\partial A^{(k)}(x')}{\partial x'^{\nu}}\right]d\sigma'^{\nu}\,U[\sigma] \qquad \text{(I-34)}$$

$$(\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} + m_{\ell})\,\underset{\sim}{\psi}^{(\ell)}(x) = i\,U^{-1}[\sigma]\int_\sigma\left[\mathcal{H}(x),\,\gamma^{\nu}\psi^{(\ell)}(x')\right]\,d\sigma'_{\nu}\,U[\sigma] \qquad \text{(I-35)}$$

Now as $\displaystyle\frac{\delta}{\delta\sigma(x')}\int_\sigma\left[\mathcal{H}(x),\,\frac{\partial A^{(k)}(x')}{\partial x'_{\nu}}\right]d\sigma'_{\nu} = \left[\mathcal{H}(x),\Box' A^{(k)}(x')\right] = 0$ for $x-x'$ spacelike, in view of (I-2) and (I-33), the condition that the second member of (I-34) will not depend on σ (as the first one) is, (by (I-32), with $F(x) = \displaystyle\int_\sigma\left[\mathcal{H}(x),\,\frac{\partial A^{(k)}(x')}{\partial x'_{\nu}}\right]\,d\sigma'_{\nu}$) :

$$\int_{\sigma(x)} d\sigma'^{\nu}\left[\mathcal{H}(x''),\left[\mathcal{H}(x),\,\frac{\partial A^{(k)}(x')}{\partial x'^{\nu}}\right]\right] = 0 \text{ if } \quad x'' \in \sigma(x), \qquad \text{(I-36)}$$

which is true if (I-33) holds.

Also we have, in view of (I-1) and (I-33):

$$\frac{\delta}{\delta\sigma(x')}\int_\sigma\left[\mathcal{H}(x),\,\gamma^{\nu}\psi^{(\ell)}(x')\right]\,d\sigma'_{\nu} = -m_{\ell}\left[\mathcal{H}(x),\psi^{(\ell)}(x')\right] = 0 \qquad \text{(I-37)}$$

If we take now $F(x) = \displaystyle\int_\sigma\left[\mathcal{H}(x),\gamma^{\mu}\psi^{(\ell)}(x')\right]\,d\sigma'_{\mu}$ the second member of (I-35) also will not depend on σ if:

$$\int \left[\mathcal{H}(x''), \left[\mathcal{H}(x), \gamma^{\nu} \psi^{(t)}(x') \right] \right] d\sigma'_{\mu} = 0 \quad \text{if } x, x'' \in \sigma(x). \qquad \text{(I-38)}$$

This is again assured by (I-33), which also assures both the integrability of Schwinger-Tomonaga equation (I-3) and the conservation of energy-momentum. No attempt of generalization of the condition (I-33), as well as of (I-25), has yet been made. Possibly this cannot be done in the scheme of local field theories.

3). Schrödinger representation:

We shall call Shrödinger representation that representation in which:

a) The field operators do not depend explicitly on time.

b) The time dependence of the wave function includes that which comes from the fact that the operators are time independent as well as that which comes from the fact that the surfaces σ are to be restricted to those on which t = constant.

In order to do this, starting from interaction representation, we first take:

$$i \frac{d\Psi[t]}{dt} = \int i \frac{\delta \Psi[t]}{\delta t(x)} d^3x = \int_t d^3x \, \mathcal{H}(\vec{x},t) \Psi[t] \qquad \text{(I-39)}$$

Now we make the usual canonical transformation:

$$\Psi'[t] = e^{-iH_{(o)}t} \Psi(t) \qquad \text{(I-40)}$$

$$F(\vec{x}) = e^{+iH_{(o)}t} F(x) e^{-iH_{o}t} \qquad \text{(I-41)}$$

$$F(x) = e^{-iH_{(o)}t} F(\vec{x}) e^{iH_{o}t} \qquad \text{(I-42)}$$

where $\vec{x}$ represent the space coordinates and $H_{(o)} = -P_o^{(o)}$ taken at t = 0;

$\Psi(t)$ and $F(\vec{x})$ are, respectively, the new wave function and the new operators. Of course $F(\vec{x})$ is equal to $F(x)$ at $t = 0$.

It should be observed, in connection with the definition of $H_{(o)}$, that the time derivatives of field operators appearing in the expression of $P^{(o)}(t)$ should be eliminated by the use of the equations of motion (I-1) and (I-2), before making $t = 0$ (there may be some difficulties in some special cases: this is not, however the situation in the cases we shall be concerned with).

In view of the transformation (I-40) the new wave function $\Psi(t)$ satisfies the Schrodinger equation:

$$i \frac{d \Psi(t)}{dt} = H \Psi(t) \qquad \text{(I-43)}$$

with:

$$H = H_{(o)} + \int d_3 x \, \mathcal{H}(\vec{x}) \qquad \text{(I-44)}$$

In this representation the relativistic covariance of the equations is not as obvious as in the two preceding ones, as time and space coordinates are not treated on the same footing.

4) S Matrix formalism.

If the wave function $\Psi[\sigma]$ satisfies a Tomonaga-Schwinger equation:

$$i \frac{\delta \Psi[\sigma]}{\delta \sigma(x)} = \mathcal{H}(x) \Psi[\sigma] \qquad \text{(I-45)}$$

and if we know its expression $\Psi[\sigma']$ at a given space-like surface σ', then we can express it, formally, for any other spacelike surface σ as:

$$\Psi[\sigma] = S[\sigma \sigma'] \Psi[\sigma'] \qquad \text{(I-46)}$$

where $S[\sigma \sigma']$ is a solution of the equation:

$$i \frac{\delta S[\sigma \sigma']}{\delta \sigma(x)} = \mathcal{H}(x) \, S[\sigma \sigma'] \qquad \text{(I-47)}$$

satisfying the boundary condition:

$$S[\sigma\sigma] = 1 \qquad (I-48)$$

Also $S[\sigma\sigma']$ should be an unitary matrix:

$$S[\sigma\sigma']^{\dagger} = S[\sigma\sigma']^{-1} \qquad (I-49)$$

as is necessary for the conservation of probability; here the unitarity of $S[\sigma\sigma']$ is a consequence of the fact that $\mathcal{H}(x)$ in equation (I-47) is hermitian.

Finally, as $\Psi[\sigma']$ should also be expressed in terms of $\Psi[\sigma]$ as:

$$\Psi[\sigma'] = S[\sigma'\sigma]\,\Psi[\sigma] \qquad (I-46a)$$

we find that:

$$S[\sigma'\sigma] = S[\sigma\sigma']^{-1} = S[\sigma\sigma']^{\dagger} \qquad (I-50)$$

Thus we can add to equation (I-47)

$$i\,\frac{\delta S[\sigma'\sigma]}{\delta\sigma(x)} = -S[\sigma'\sigma]\,\mathcal{H}(x) \qquad (I-47a)$$

Another important property of the matrices $S[\sigma\sigma']$ is expressed by the relation:

$$S[\sigma\sigma'] = S[\sigma\sigma_1]\,S[\sigma_1\sigma'] \qquad (I-51)$$

where σ_1 is an <u>arbitrary</u> space-like surface.

In order to prove relation (I-51) we first prove that its second member is independent of σ_1. For this it is sufficient to notice that in consequence of (I-47) and (I-47a) we find:

$$\frac{\delta}{\delta\sigma_1(x)}\left\{S[\sigma\sigma_1]\,S[\sigma_1\sigma']\right\} = \frac{\delta S[\sigma\sigma_1]}{\delta\sigma_1(x)}\,S[\sigma_1\sigma'] + $$
$$+ S[\sigma\sigma_1]\,\frac{\delta S[\sigma_1\sigma']}{_1(x)} = 0 \qquad (I-52)$$

Indeed if we start with a given intermediate surface σ_1 in the expression on the second member of (I-51) and change continuously this surface until any other spacelike surface is obtained, we see that the final expression is equal to the initial one as all variations are of the type of (I-52), and thus vanishing.

Now if we take in special $\sigma_1 = \sigma$ or $\sigma_1' = \sigma'$, then we see that the second member of (I-52) becomes equal to $S[\sigma\sigma']$, as stated in the first member.

An immediate consequence of the property expressed in expression (I-51) is that it permits to express the finite transformation $S[\sigma\sigma']$ as a product of infinitesimal transformations ordered along <u>any</u> path (of ordered space-like surfaces) from σ' to σ:

$$S[\sigma\sigma'] = S[\sigma\sigma_n]\, S[\sigma_n\, \sigma_{n-1}] \cdots\cdots\, S[\sigma_2\sigma_1]\, S[\sigma_1\sigma'] \quad \text{(I-53)}$$

The expression of the infinitesimal transformation $S[\sigma_{i+1}, \sigma_i]$ is, from equation (I-47) given by:

$$S[\sigma_{i+1}, \sigma_i] = 1 - i\int_{\sigma_i}^{\sigma_{i+1}} d_4x\, \mathcal{H}(x) \,. \quad \text{(I-54)}$$

Dyson[6] has shown that using (I-54) we can re-write (I-53), once the ordering of the surfaces is given, such that only one surface passes by each point, as:

$$S[\sigma\sigma'] = 1 - i\int_{\sigma'}^{\sigma} \mathcal{H}(x)\, d_4x - \int_{\sigma'}^{\sigma}\mathcal{H}(x) d_4x \int_{\sigma'}^{\sigma(x)} \mathcal{H}(x')\, d_4x' + \cdots \quad \text{(I-55)}$$

where $\sigma(x)$ is the one of such surfaces which passes by the point x. The fact that expression (I-55) is independent of the parametrization of the surfaces is a consequence of the same property valid for (I-53), as shown before.

It is easy to verify that the same expression (I-55) still holds formally in the case when σ and σ' do intersect each other, as long as we parametrize the region between these surfaces with the help of an ordered family of space-like surfaces passing by the intersection of σ and σ'.

It should be observed that the possibility of assuming expression (I-55) as valid for <u>any surfaces σ and σ'</u> is a consequence of the integrability of equation (I-45), which is only true if the integrability condition:

$$\left[\mathcal{H}(x), \mathcal{H}(x')\right] = 0 \text{ , if } x-x' \text{ is space-like,} \qquad (I-56)$$

is satisfied. This is also essentially the condition necessary for the independence of expression (I-55) for $S[\sigma\sigma']$ on the parametrization of the surfaces. If we want to try an S Matrix formulation of a theory for which condition (I-56) is not satisfied (as will be considered in the next section) then we should restrict the definition of the S Matrix to an special group of surfaces in such a way that the parametrization will be unique in each case (say, plane spacelike surfaces).

Heisenberg - Stueckelberg - Feynman S Matrix:

If we know the wave function $\Psi[\sigma_{-\infty}]$ at the surface in the infinite past then the wave function at any space-like surface $\Psi[\sigma]$ will be given by:

$$\Psi[\sigma] = S[\sigma]\ \Psi[\sigma_{-\infty}] \qquad (I-57)$$

where: $S[\sigma] = S[\sigma\ \sigma_{-\infty}]$, and the wave function at the infinite future by:

$$\Psi[\sigma_{+\infty}] = S\ \Psi[\sigma_{-\infty}] \qquad (I-58)$$

where: $S = S[\sigma_{+\infty}, \sigma_{-\infty}]$

The matrix S can be written, according to Dyson[6] as:

$$S = \sum_{n=0}^{\infty} (-i)^n \frac{1}{n!} \int_{-\infty}^{+\infty} \cdots \int d_4x_1 \ldots d_4x_n \; P\,(\mathcal{H}(x_1) \ldots$$

$$\ldots \mathcal{H}(x_n)) \qquad (I-59)$$

where P is an operator which orders the terms in the bracketed product according to the ordering of the surfaces $\sigma(x_1) \ldots \sigma(x_n)$. Now, for the usual types of theories, the matrix elements, between free states, of the expressions $P\,(\mathcal{H}(x_1) \;\ldots\; \mathcal{H}(x_n))$ can be written in terms of the $S_F(x-x')$ and $\Delta_F(x-x')$ functions (c-numbers) used by Feynman[3] and Stueckelberg[4] and thus the matrix S is shown to coincide with the S-Matrix used by them in configuration representation.

Invariance of the S-Matrix formalism:

There are three conditions which should be satisfied by the S-Matrix (besides those considered of unitarity and casuality), which in the present case are a direct consequence of the fact that the theory is derived from an invariant Tomonaga-Schwinger equation:

1) Form invariance.

2) Equivalence of any set of ordered σ surfaces.

3) Independence of the Matrix $S = \lim_{\substack{\sigma' \to \sigma_{-\infty} \\ \sigma \to \sigma_{+\infty}}} S\,[\sigma, \sigma']$ from the limiting surfaces $\sigma_{+\infty}$, $\sigma_{-\infty}$.

The first condition is here fulfilled as a consequence of the fact that $\mathcal{H}(x)$ is a scalar:

If we make a Lorentz Transformation:

$$x^\mu \longrightarrow x'^\mu = a^\mu{}_\nu \, x^\nu \qquad (I-60)$$

then the wave function will undergo, in general (an analysis of the

Lorentz transformation will be made in the part II), an unitary transformation:

$$\Psi[\sigma] \longrightarrow U\,\Psi[\sigma] \tag{I-61}$$

such that, if $\mathcal{H}(x)$ is a scalar we will have:

$$U\,\mathcal{H}(x)\,U^{-1} = \mathcal{H}(x'), \tag{I-62}$$

the unitary (in general) operator U being determined (Part II) in such a way that the free field equations remain invariant.

It is a consequence of the scalar nature of $\mathcal{H}(x)$ expressed by (I-62) that the expression

$$U\,S[\sigma, \sigma']\,U^{-1}$$

corresponding to the S matrix in the new reference frame, for the <u>same surfaces σ, σ' and the same parametrization</u> has the same form as $S[\sigma, \sigma']$, in the original system.

The second condition can be expressed more precisely in the following way:

Consider two families of space-like surfaces ordered from the infinite past to infinite future: σ_1, σ_2, σ_3, ... and σ'_1, σ'_2, σ'_3, ..., and the correspondence:

$$\sigma_i \longleftrightarrow \sigma'_i \, . \tag{I-63}$$

thus there should be an unitary transformation connecting the wave function in the corresponding surfaces:

$$\Psi[\sigma_i] = U[\sigma_i\,\sigma'_i]\,\Psi[\sigma'_i] \tag{I-64}$$

and also there should be a relation between the corresponding S matrices:

$$S[\sigma_i\,\sigma_j] = U[\sigma_i\,\sigma'_i]\,S[\sigma'_i\,\sigma'_j]\,U[\sigma'_j\,\sigma_j] \tag{I-65}$$

Now in view of the previous analysis it is clear that (I-64) and (I-65) are verified if we take:

$$U\left[\sigma_i\ \sigma_i'\right] = S\left[\sigma_i\ \sigma_i'\right] \qquad \text{(I-66)}$$

It is possible that in an S-Matrix theory not derivable from a Tomonaga-Schwinger equation we could have expressions of the type (I-64) and (I-65) with an U matrix essentially different from the S matrix.

The third condition referred to above, that the matrix $S = S\left[\sigma_{+\infty}\ \sigma_{-\infty}\right]$ be independent of the two limiting surfaces, is a consequence of the fact that we impose, at least for scattering processes, that $\Psi\left[\sigma_{-\infty}\right]$ and $\Psi\left[\sigma_{+\infty}\right]$ be independent of the special surface involved:

$$\frac{\delta\, \Psi[\sigma]}{\delta\, \sigma(x)} = 0 \quad \text{for} \quad \sigma = \sigma_{-\infty} \text{ or } \sigma_{+\infty} \qquad \text{(I-67)}$$

Here if we consider two different families of surfaces, say of planes, and compare the expression of S obtained from (I-59) for these cases we see that the only difference coming about is the one in the order of some operators in points with space-like connection; however, in the present case this different order is unimportant in view of the commutation relation:

$$\left[\mathcal{H}(x)\ ,\ \mathcal{H}(x')\right] = 0 \quad \text{if } x - x' \text{ is space-like.} \qquad \text{(I-66)}$$

Thus the two expressions of S, for the two parametrizations, are identical.

B - Non local Schrodinger projection theories.

As said before we are mainly interested on a theory which is obtained from that used by Fireman (this will be analyzed in detail in Parts III and IV) by the substitution:

$$\psi(x) \longrightarrow \psi_+(x) \quad \text{for the neutrino field,} \qquad \text{(I-68)}$$

where $\psi_+(x)$ is the positive energy part of $\psi(x)$ we shall call a theory resulting from any usual one by a substitution of the type (I-68), a "Schrodinger projection theory".)

In view of the substitution (I-68) the interaction hamiltonian $\mathcal{H}(x)$ of the original theory goes into a new expression $\tilde{\mathcal{H}}(x)$ for which, in general, $\left[\tilde{\mathcal{H}}(x), \tilde{\mathcal{H}}(x')\right] \neq 0$ even for x-x' spacelike (I-69)

Thus we will not be able to write, anymore, a Tomonaga-Schwinger equation, although we can still express the resulting theory in Schrodinger representation, where the Hamiltonian H in equation (I-45) is given by:

$$H = H_{(o)} + \int \tilde{\mathcal{H}}(\vec{x})\, d_3x \qquad (I\text{-}70)$$

In the case of Fireman's theory(10) the (Fermi) interaction used is, in the scalar case, given by:

$$\mathcal{H}(x) = g\,\overline{\psi}_p(x)\psi_N(x)\,\overline{\psi}_e(x)\left[\psi_\nu(x) + \gamma^5 C\,\overline{\psi}_\nu(x)\right] + \text{h.c.} \qquad (I\text{-}71)$$

where "h.c." means the hermitian conjugate of the preceding terms. By the Schrodinger projection (I-68) we obtain:

$$\tilde{\mathcal{H}}(x) = g\,\overline{\psi}_p(x)\psi_N(x)\,\overline{\psi}_e(x)\left[\psi^+_\nu(x) + \gamma^5 C\,\overline{\psi}^+_\nu(x)\right] + \text{h.c.} \qquad (I\text{-}72)$$

which, as it is of immediate verification, is of the type (I-69).

This resulting theory, formulated in Schrodinger representation, is essentially the one used by Fireman when, in the application of the theory given by (I-71) to double β - decay, he neglected the antineutrino contributions to the intermediate states.

However, we should analyse now the relativistic invariance of such a theory. The Schrodinger representation being not appropriate to the analysis of the relativistic invariance we first formulate this theory in the S - matrix representation.

Here we define:

$$S[t] = 1 - i \int_{-\infty}^{t} \tilde{\mathcal{H}}(x)\, d_4x - \int_{-\infty}^{t} \tilde{\mathcal{H}}(x)\, d_4x \int_{-\infty}^{t(x)} \tilde{\mathcal{H}}(x')\, d_4x' + \cdots$$

$$+ \cdots\cdots, \qquad \text{(I-73)}$$

in analogy to (I-55), where $t(x)$ is a plane, normal to the time direction, passing by the point x.

Now it is seen that if we define:

$$\Psi[t] = S[t]\, \Psi[-\infty] \qquad \text{(I-74)}$$

then $\Psi[t]$ will satisfy the equation (I-39), which was shown to be equivalent to the Schrodinger equation. The S-matrix will be then:

$$S = \lim_{t \to \infty} S[t]$$

Now we compare the expressions of S obtained by two observers whose reference frames have different time directions, which should be identical in view of the third condition for invariance of the S-matrix formalism. Here, as before (equation I-59), we can express S as:

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} \cdots \int d_4x_1 \cdots d_4x_n\, P\left(\tilde{\mathcal{H}}(x_1) \cdots \tilde{\mathcal{H}}(x_n)\right) \qquad \text{(I-75)}$$

Now, as the ordering operator P is not the same for the two coordinate systems, we see that some factors in the integrands of (I-75) which will appear for one system in the order $\tilde{\mathcal{H}}(x)\,\tilde{\mathcal{H}}(x')$, will appear in the second case as $\tilde{\mathcal{H}}(x')\,\tilde{\mathcal{H}}(x)$ for some $x-x'$ spacelike. Now in view of (I-69) these contributions will be different and thus the two expressions for S will not be in general identical as they should be. However, we should analyse every case separately as compensations might occur in some cases.

In the special case when $\tilde{\mathcal{H}}(x)$ is of the type (I-72) we can evaluate from (I-75), following Dyson's method, the Feynman term corresponding to the diagram in which a neutrino is exchanged between two incident neutrons which decay into two protons and two electrons. We find that, instead of the usual invariant Feynman function for the neutrino line, the function will appear:

$$\varepsilon(x_1 - x_2) \quad S_F(x_1 - x_2) \qquad \text{(I-76)}$$

which is not invariant in view of the factor $\varepsilon(x_1 - x_2)$ defined by:

$$\varepsilon(x) \quad = \quad \text{sign}\; x_o \qquad \text{(I-77)}$$

The same conclusion would be reached by the actual computation of the same process (double β decay of two colliding neutrons) from the Schrodinger equation.

It should be observed here that not all of the Schrodinger projection theories are not relativistically invariant. In some special cases, as will be considered in Parts III and IV the new $\tilde{\mathcal{H}}(x)$ obtained by the substitution (I-58) still satisfies the integrability condition (I-56) and the theory is, therefore, relativitically invariant.

PART II

INVARIANCE OF FIELD THEORY

UNDER IMPROPER LORENTZ TRANSFORMATIONS

This chapter is concerned primarily with restrictions on β decay and meson interactions which can be obtained by studying the transformation laws of the field quantities under the improper Lorentz group. The proper Lorentz group will also be discussed to some extent in section A, mainly for the purpose of illustrating how one is to formulate the problem of invariance of the wave equations[11]. The antiunitary transformation suggested by Wigner[12] for time inversion is also introduced in the form which corresponds to the transformation of the one particle wave function given by Newton and Wigner[13].

In section B the problem of invariance of Quantum Electrodynamics to time inversion is investigated. Two possible transformations are set up which can represent time inversion. The first of these, denoted by I, is the one already introduced in section A, and it is characterized by the property that the spatial part of the electromagnetic vector potential reverts its sign under time inversion while the time-like part does not. The second transformation, denoted by II, has the property that the vector potential transforms like an ordinary vector under time inversion.

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11). We follow, in section A, the lines of a forthcoming paper on the subject of Lorentz invariance by Wigner and Wightman, inasmuch as we know of it from discussions with the authors.

12). E. P. Wigner, Gottinger Nachrichten, p. 546, 1932.

13). T. D. Newton and E. P. Wigner, Rev. Mod. Phys. 21, 400, 1949.

In section C, it is found that two more transformations (denoted respectively by III and IV) are possible, but only for the special case of zero rest mass. Thus, III and IV may be applied for the case of the neutrino, but not for any particle with finite rest mass. There is no reason in principle why all spinor fields should transform in the same way under time inversion. This problem is therefore investigated here, and the possible combinations of fields transforming according to I, II, III and IV are analyzed in detail. We pay special attention to the Fermi type of interaction in this study.

In section D, an investigation is made of the possibility of introducing additional phase factors in the improper transformations. The possible values of the phase factors are found and a discussion is given on the bearing of the transformation properties of spin 1/2 fields on the symmetry of the π-meson field function. Various types of β-decay interactions are then discussed with regard to their behaviour under improper Lorentz transformations. This problem is of interest because, without the use of the phase factors, several of these possibilities would not lead to invariance of the Hamiltonian under the improper Lorentz transformations. It is shown, however, that with an appropriate choice of these phase factors all of the referred interactions can be made invariant, so that they each provide a possible basis for a consistent theory of β-decay (this point will be investigated further in Part IV). Finally, the transformation properties under the improper Lorentz group of a Majorana field are studied.

A. Preliminary concepts and results.

For simplicity of language it is convenient to introduce the concepts of bodily identity and subjective identity[14]. A given physical system (or several equivalent systems) can be viewed by several observers using different coordinate frames. The system and its state, although in different relations to the several observers, is then said to be bodily identical for all of them. On the other hand two different systems, each observed by a different observer, are said to be subjectively identical if they each bear the same relation, to the corresponding observers. In what follows we shall interpret the Lorentz transformation as a passive transformation, i.e., a change in the coordinate system given by:

$$x^{\mu} \longrightarrow \tilde{x}^{\mu} = a^{\mu}{}_{\nu}\, x^{\nu} \qquad \text{(II-1a)}$$

with

$$a^{\mu}{}_{\nu}\, a_{\mu\lambda} = g_{\nu\lambda} \qquad \text{(II-1b)}$$

The tilde will be used to indicate the quantities (coordinates, wave functions, operators) used by the new observer.

As a consequence of transformation (II-1) the wave function Ψ, observable quantities and auxiliary field quantities (ψ, A) will be in general also transformed. The new quantities (with tilde) refer, of course, to the bodily identical system. We shall use here the usual condition for relativistic invariance: The invariance of the form of the equations of motion.

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14). A. S. Wightman and E. P. Wigner, forthcoming paper. The form in which some of their concepts and results are reproduced here, with the kind permission of the above authors, is not necessarily the same as that which they will present in their paper.

Among the many possible formulations of the Lorentz Transformation in Quantum Theory there are two especially simple, the Heisenberg and the Schrödinger[14]. These derive their name as a result of their similarity with the corresponding representations of the Field Theory.

1). Schrödinger type of Lorentz transformation.

Here the two observers use the same wave function for two subjectively identical systems (and thus, different wave functions for bodily identical systems). On the other hand both observers will work with the same set of operators.

Thus we have the transformation:

$$\Psi[\sigma] \longrightarrow \tilde{\Psi}[\sigma] = U\,\Psi[\sigma] \qquad \text{(II-2a)}$$

where U is in general a unitary operator (we exclude here the anti-unitary case of time inversion, which we consider in section 4).

U will be chosen in such a way that we have:

$$U\,\psi(x)\,U^{-1} = \Lambda^{-1}\,\psi(x') \qquad \text{(II-2b)}$$

for spinor fields ψ, where the operator Λ (acting on the spinor space) is such that:

$$\Lambda\,\gamma^{\mu}\,\Lambda^{-1} = a^{\mu}{}_{\nu}\,\gamma^{\nu} \quad (15) \qquad \text{(II-2c)}$$

and:

$$U\,A_{\mu}(x)\,U^{-1} = a_{\mu}{}^{\nu}\,A_{\nu}(x') \qquad \text{(II-2d)}$$

for vector fields; the generalization to other types of fields is obvious.

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15). For the form of the operator Λ for the several types of Lorentz transformation, see for instance: W. Pauli, Handbuck der Physik, Vol.241, p. 259; Rev. Mod. Phys. 13, 203, 1941.

2). Heisenberg type of Lorentz transformation.

In this case the same wave function is used by the two observers for bodily identical systems, say, for the same physical system as viewed by both of them, although they will use different observables and auxiliary field quantities. The transformations are now, instead of (II-2),

$$\Psi[\sigma] \longrightarrow \tilde{\Psi}[\sigma] = \Psi[\sigma] \qquad \text{(II-3a)}$$

$$\psi(x) \longrightarrow \tilde{\psi}(\tilde{x}) = \Lambda\psi(x) \qquad \text{(II-3b)}$$

$$A_\mu(x) \longrightarrow \tilde{A}_\mu(\tilde{x}) = a^\nu{}_\mu A_\nu(x) \qquad \text{(II-3c)}$$

The justification of the last two transformations is as follows. The equivalence of the Heisenberg and Schrödinger forms of Lorentz transformation imposes the equality of the expectation values of corresponding quantities:

$$(\Psi[\sigma], \tilde{F}(\tilde{x})\, \Psi[\sigma])_H = (\tilde{\Psi}[\sigma], F(\tilde{x})\, \tilde{\Psi}[\sigma])_S \qquad \text{(II-4)}$$

where the indices H and S appear only to indicate to us that we use, respectively, the Heisenberg and Schrödinger transformed quantities. From (II-4) there results

$$\tilde{F}(\tilde{x}) = U^{-1} F(\tilde{x})\, U \ , \qquad \text{(II-5)}$$

which was used in obtaining (II-3b,c) from (II-2).

It should be observed that in the Heisenberg form of the Lorentz transformation the spinor field quantities $\psi(x)$ transform in the same way as the wave functions $\psi(x)$ in the one particle Dirac equation[15].

3). Relation between the transformation (II-2b) for the field operator $\psi(x)$ and the transformation of the one particle Dirac wave function $\psi(x)$.

In order to justify the transformation (II-2b) which we assumed as valid in the Schrödinger type of Lorentz transformation we should verify that it leads to the usual transformation for the one particle Dirac wave function $\varphi(x)$ in configuration space:

$$\varphi(x) \longrightarrow \tilde{\varphi}(\tilde{x}) = \Lambda(x) \qquad \text{(II-6)}$$

This can be done in the following way. We first express the quantum wave function Ψ for a system with one particle (say, electron) as:

$$\Psi = \int_{\sigma} d\sigma_{\mu} \overline{\Psi_{+}}(x) \gamma^{\mu} \varphi(x) \Omega_{0} \qquad \text{(II-7)}$$

where $\overline{\Psi_{+}}(x)$ is the creation operator for electrons and Ω_0 is the vacuum wave function, and:

$$\Psi_{+}(x) \Omega_{0} = 0. \qquad \text{(II-8)}$$

Now, if we write the new wave function in the same form as (II-7):

$$\tilde{\Psi} = \int_{\tilde{\sigma}} d\tilde{\sigma}_{\mu} \overline{\Psi_{+}}(\tilde{x}) \gamma^{\mu} \tilde{\varphi}(\tilde{x}) \tilde{\Omega}_{0} \qquad \text{(II-9)}$$

we see, by application of (II-2) to (II-7) and comparison with (II-9), that the positive energy wave function $\varphi(x)$ transforms according to (II-6). The extension of this method to integer spin fields is obvious.

4). Time inversion.

In addition to the requirements of invariance of our field theories under the proper, restricted, Lorentz group we require also invariance under the improper Lorentz Group.

The requirement of invariance under space reflexions leads to the well known, and very useful, concept of parity. The condition of invariance under time inversion is equivalent to the assumption of the Principle of Microscopic Reversibility[14].

For the transformation, by time inversion, of the one particle Dirac wave function in configuration representation we adopt Wigner's anti-unitary transformation[12), 13)]

$$\varphi(x) \longrightarrow \breve{\varphi}(\breve{x}) = \Lambda \varphi^{*}(x) = \gamma_5 C \varphi^{*}(x) \quad , \qquad \text{(II-10)}$$

which transforms positive energy wave functions into positive energy ones. The matrix C in (II-10) is defined by:

$$C \gamma^{\mu} C = - \gamma^{\mu}_{T} \qquad \text{(II-10a)}$$

$$C^{\dagger} = C^{-1} = - C_{T} = C \qquad \text{(II-10b)}$$

where the indice T is used to indicate the transposed matrix. Now, by the method described in the subsection 3) we find, for the Schrödinger type of transformation in Field Theory that, assuming

$$\Psi[\sigma] \longrightarrow \widetilde{\Psi}[\sigma] = U \Psi^{*}[\sigma] \quad , \qquad \text{(II-11a)}$$

the field operator $\psi(x)$ should transform as:

$$U \psi^{*}(x) U^{-1} = \Lambda^{-1} \psi(\breve{x}) \quad , \qquad \text{(II-11b)}$$

with: $$\Lambda = \gamma_5 C \qquad \text{(II-11c)}$$

In (II-11b) $\psi^{*}(x)$ is the operator which is represented by the complex conjugate (not hermitian conjugate) of the matrix which corresponds to $\psi(x)$ in the same representation.

Now, in order to find the Heisenberg form of the time-inversion transformation, equivalent to (II-1) we impose again condition (II-4) for the invariance of expectation values and find, instead of (II-5):

$$\widetilde{F}(\breve{x}) = \left[U^{-1} F(x) U \right]_{t} \qquad \text{(II-12)}$$

where the index t means the transposed of the operator in the bracket.

(In other words, F_t is the operator which is represented by the transposed matrix of that which represents F).

Thus we find, for the Heisenberg form of Lorentz transformations:

$$\Psi[\sigma] \longrightarrow \tilde{\Psi}[\sigma] = \Psi[\sigma] \qquad \text{(II-13a)}$$

$$\psi(x) \longrightarrow \tilde{\psi}(\check{x}) = \Lambda\psi^{\dagger}(x) = \beta\gamma_5 C\,\overline{\psi}(x) \qquad \text{(II-13b)}$$

The transformation (II-13b), for the field operator $\psi(x)$, is formally the same as (II-10) for the wave function $\varphi(x)$, as in the case of unitary transformations. However, in the case of transformations (II-13) there is an additional operation, coming from the application of (II-12), in the case when F is a product of field operators:

$$F = F_1 F_2 \cdots F_n \qquad \text{(II-14)}$$

Then, in view of (II-12) we find:

$$F = (U^{-1} F_1 U \cdot U^{-1} F_2 U \cdots U^{-1} F_n U) =$$

$$= (\tilde{F}_1^{\,t}\, \tilde{F}_2^{\,t} \cdots \tilde{F}_n^{\,t})_t$$

or:

$$\tilde{F} = \tilde{F}_n \cdots \tilde{F}_2\, \tilde{F}_1 \qquad \text{(II-15)}$$

Thus we have to add to transformations (II-13) the operation[16)]

$$F_1(x)\, F_2(x) \cdots F_n(x) \longrightarrow \tilde{F}_n(\check{x}) \cdots \tilde{F}_2(\check{x})\, \tilde{F}_1(\check{x}) \qquad \text{(II-13c)}$$

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16). This rule of reverting the order of the field operators in connection with a time inversion was first obtained by K. M. Case, by imposing the invariance of the commutation relations. This was the result which lead him to the conclusions referred to in the appendix of his paper in Phys. Rev. 76, 1, 1949, which we shall analyse later (private communication).

For instance:

$$\overline{\psi}(x)\ \psi(x) \longrightarrow \tilde{\psi}(x)\ \tilde{\overline{\psi}}(\tilde{x})$$

From now on we shall refer to the time inversion transformation (II-13) as transformation I. Other forms of transformation will be considered in the following sections.

B. Invariance of Quantum Electrodynamics under time inversion.

It is simpler to make this analysis in the Heisenberg representation, in which the equations of motion of the electron field $\psi(x)$ and electromagnetic field $A_\mu(x)$ are the following[2]:

$$\left[\gamma^\mu \left(\frac{\partial}{\partial x^\mu} + i\,e\,A_\mu(x)\right) + m\right]\psi(x) = 0 \qquad \text{(II-16)}$$

$$\Box\, A_\mu(x) = j_\mu(x) \qquad \text{(II-17)}$$

with:

$$j_\mu(x) = \frac{ie}{2}\left[\overline{\psi}(x)\,\gamma_\mu\,\psi(x) - \psi(x)\,\gamma_\mu^T\,\overline{\psi}(x)\right] \qquad \text{(II-17a)}$$

$$\frac{\partial A^\mu(x)}{\partial x^\mu}\,\Phi = 0 \qquad \text{(II-18)}$$

These equations should be invariant under the Lorentz transformation (we restrict ourselves from now on to the Heisenberg form of Lorentz transformation). This is the case if we add to the transformation (II-13b) for $\psi(x)$:

$$\psi(x) \longrightarrow \tilde{\psi}(\tilde{x}) = \beta\,\gamma_5\,C\,\overline{\psi}(x), \qquad \text{(II-19a)}$$

the following transformation for $A_\mu(x)$:

$$A_0(x) \longrightarrow \tilde{A}_0(\tilde{x}) = A_0^\dagger(x) = A_0(x)\ ;\ \vec{A}(x) \longrightarrow -\vec{A}(x) \qquad \text{(II-19b)}$$

To these should be added equations (II-13a):

$$F_1(x)\, F_2(x)\, \ldots\, F_n(x) \longrightarrow \tilde{F}_n(\tilde{x})\, \ldots\, \tilde{F}_2(\tilde{x})\, \tilde{F}_1(\tilde{x}) \qquad \text{(II-19c)}$$

It should be observed that transformation (II-19b) differs from the usual one for a covariant vector by a sign. The same is also true for the transformation of $j_\mu(x)$, thus making $j^\mu(x)\, A_\mu(x)$ an scalar density. Therefore the quantity $\frac{\partial A^\mu}{\partial x^\mu}$ is not invariant in a time inversion, as it changes sign. However, this is unimportant as this last quantity has always an expectation value zero, in view of the auxiliary condition (II-18).

Equation (II-19a) should better be written:

$$\psi_+(x) \longrightarrow \beta\gamma_5\, C\, \overline{\psi}_+(x) \qquad \text{(II-20a)}$$

$$\psi_-(x) = C\psi'_+(x) \longrightarrow \beta\gamma_5 C\, \overline{\psi}_-(x) = \beta\gamma_5 \psi'_+(x) \qquad \text{(II-20b)}$$

$$\psi'_+(x) = C\, \overline{\psi_-}(x) \longrightarrow \beta\gamma_5\, C\overline{\psi'}_+(x) \qquad \text{(II-20c)}$$

where

$$\psi'(x) = C\, \overline{\psi}(x)$$

is the charge conjugate field.

In transformations (II-20) $\psi_+(x)$ and $\psi'_+(x)$ are, respectively, operators of absorption of electrons and positrons. We see from these transformations that:

1) Positive energy operators go into positive energy ones.

2) Electrons (or positrons) go into electrons (or positrons).

3) Absorption operators go into emission operators and vice-versa.

Now, we see that, besides the transformation (II-20) for time inversion, which is the one that we have called <u>transformation I</u> in sec. A, another type of time inversion exists, such as:

$$\psi_+(x) \longrightarrow \beta\gamma_5 \, C \, \overline{\psi}'_+(x) \tag{II-21a}$$

$$\psi_+(x) \longrightarrow \beta\gamma_5 \, C \, \overline{\psi}_+(x) \tag{II-21b}$$

or, instead of (II-19a)

$$\psi(x) \longrightarrow \tilde{\psi}(\tilde{x}) = \beta\gamma_5 \, C \, \overline{\psi}^{a}(x) \tag{II-22a}$$

Now the equations of motion (II-16, 18) will be invariant if we add to (II-22a), instead of (II-19b):

$$A_0(x) \longrightarrow -A_0(x) \; ; \; \vec{A}(x) \longrightarrow \vec{A}(x) \tag{II-22b}$$

and keep (as is necessary[17]) equation (II-19c):

$$F_1(x)\, F_2(x) \longrightarrow \tilde{F}_2(\tilde{x}) \; \tilde{F}_1(\tilde{x}) \tag{II-22c}$$

We should observe that now the transformation (II-22b) is the usual one for covariant vectors. We shall refer to the transformation (II-22), for time inversion as <u>transformation II</u>. In this case we see, from (II-21) or (II-22a) that positive energy operators go into positive energy operators and absorption operators into emission ones, as in the case of transformation I. However, now, electron operators go into positron operators and vice-versa.

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17). This is because in the Schrödinger form of the Lorentz transformation we use again an anti-unitary transformation of the type (II-11a), as is necessary in order that positive energy electron (positron) wave functions in the one particle configuration space will go into positive energy positron (electron) wave function.

If we notice that (II-22a) can be written as:

$$\psi(x) \longrightarrow \beta \gamma_5 \psi(x) \tag{II-23}$$

We see that this type of time inversion is formally the same as the one given by Pauli[15], in the c-number theory, except for the additional operation (II-22c). This operation is, however, fundamental in the q-number theory as the anticommutation of the operators $\psi(x)$ and $\overline{\psi}(x)$ would produce otherwise a change of sign in the second member of (II-17) which destroys the invariance to time inversion.

The classical analogy of these two possible types of transformation, (II-19) and (II-22), for time inversion is well known:

The classical equations for the electro-magnetic field and for the charged particle are:

$$\frac{\partial F^{\mu}{}_{\nu}(x)}{\partial x^{\mu}} = e \int_{-\infty}^{+\infty} \frac{dy_{\nu}}{ds}\, \delta(x-y)\, ds \tag{II-24}$$

$$m \frac{d^2 x^{\mu}}{ds^2} = e\, F^{\mu}{}_{\nu}(x) \frac{dx^{\nu}}{ds} \tag{II-25}$$

where:

$$F_{\mu\nu} = \frac{\partial A_{\nu}}{\partial x^{\mu}} - \frac{\partial A_{\mu}}{\partial x^{\nu}} \tag{II-26}$$

$$ds = \sqrt{-dx^{\mu}\, dx_{\mu}} \tag{II-27}$$

Now we require (as in the quantum case) that after the transformation for time inversion,

$$x_0 \longrightarrow \tilde{x}_0 = -x_0 \; ; \; \vec{x} \longrightarrow \tilde{\vec{x}} = \vec{x} \tag{II-28a}$$

the energy, which was given by $m \frac{dx^0}{ds}$ in the old system and is now $m \frac{d\tilde{x}^0}{d\tilde{s}}$, still remain positive. We must then assume that the proper time s transforms according to:

$$ds \longrightarrow d\tilde{s} = -ds \tag{II-28}$$

(or change s into $-s$).

Then the equations (II-24) and (II-25) will remain invariant in either of the following two cases:

$$\text{a)}\quad \tilde{A}_0(x') = A_0(x)\ ;\quad \vec{\tilde{A}}(x') = -\vec{A}(x),$$

In this case the charge, e, does not change sign.

$$\text{b)}\quad \tilde{A}_0(x') = -A_0(x)\ ;\quad \vec{\tilde{A}}(x') = \vec{A}(x)\ ,$$

and also change e into -e, i.e., interchange positrons and electrons.

C. Restrictions on the covariant expressions and possible interaction hamiltonians $\mathcal{H}(x)$ imposed by the invariance under time inversion.

If a given massive particle of spin 1/2 is charged, or has a magnetic moment, then the possible transformations under time inversion are the ones given by (II-19) and (II-22) which we shall refer to, from now on, as I and II, respectively (for the moment we neglect other types of interaction). This arbitrariness of choice between I and II, which exists when we have only one type of charged spinor fields disappears when there are several of them (charged or with an anomalous magnetic moment) for the following reason. The choice of I or II for one of these fields (say, the electron field) determines the transformation of the electromagnetic field (as either (II-22b) or (II-19b)) and thus all the other (charged or with anomalous magnetic moment) fields should transform in the same way as the first one (at least up to a phase factor ± 1 or $\pm i$ as will be analysed in section 4).

Now if another spinor field has a zero mass we find, besides transformations I and II, two more possibilities for the transformation under time inversion for this field. These are listed in Table I and will be referred to respectively as <u>transformations III</u> and <u>IV</u>.

TABLE I

POSSIBLE TYPES OF TRANSFORMATIONS UNDER TIME INVERSION

Name of the transformation	Transformation of the spinor field	Restriction on the mass of the spin 1/2 field	Resulting form of the transformation of the electromagnetic potential $A_\mu(x)$ and of a pseudo-scalar charged field $B(x)$, if $\psi(x)$ is a charged field.
I	$\psi \longrightarrow \beta\gamma_5 C\, \overline{\psi}$	No restriction	$A_\mu(x) \longrightarrow - \hat{A}_\mu(x)$ $B(x) \longrightarrow - B(x)^\dagger$
II	$\psi \longrightarrow \beta\gamma_5 C\, \overline{\psi'}$	No restriction	$A_\mu(x) \longrightarrow + \hat{A}_\mu(x)$ $B(x) \longrightarrow - B'(x)^\dagger$
III	$\psi \longrightarrow \beta C\, \overline{\psi}$	Zero mass	Same as for transformation II
IV	$\psi \longrightarrow \beta C\, \overline{\psi'}$	Zero mass	Same as for transformation I

In all these cases the additional reordering operation (II-22c) should be performed.

The quantity $\hat{A}_\mu(x)$ used in the fourth column of Table I is defined by:

$$\hat{A}_0(x) = - A_0(x) \; ; \; \hat{\vec{A}}(x) = \vec{A}(x). \qquad \text{(II-29)}$$

The results given in the fourth column of Table I were found as follows:

(1) The transformation of $A_\mu(x)$ should be the same as that of:

$$j_\mu(x) = \frac{ie}{2}\left[\overline{\Psi}(x)\gamma_\mu \Psi(x) - \Psi(x)\,\gamma_\mu^{T}\,\overline{\Psi}(x)\right] \quad , \quad \text{(II-17a)}$$

if $\Psi(x)$ is a charged field, in order that the wave equations (II-16), (II-17) (eventually with $m = 0$) of the field Ψ in interaction with the electromagnetic fields would be invariant[18].

(2) The transformation of a charged pseudoscalar field $B(x)$ which also interacts with the electromagnetic field was obtained by the following condition. The current density vector

$$j_\mu(x) = \frac{ie}{2}\left(B(x)^{\dagger}\,\frac{\partial B(x)}{\partial x^\mu} - B(x)\,\frac{\partial B(x)^{\dagger}}{\partial x^\mu}\right) \quad \text{(II-30)}$$

should transform in the same way as $A_\mu(x)$. Now the transformation of $A_\mu(x)$ was conditioned by that of $\Psi(x)$. Thus if both $\Psi(x)$ and $B(x)$ interact with the electromagnetic field $A_\mu(x)$ then the transformation of $B(x)$ under time invertion is conditioned by that of $\Psi(x)$ in the way given in Table I.

It should be observed that $B'(x)$ used in Table I is the charge conjugate field to $B(x)$:

$$B'(x) = B(x)^{\dagger} \quad \text{(II-31)}$$

Also, in close analogy with the corresponding transformation for $\Psi(x)$ positive energy absorption operators of the field $B(x)$ go into positive energy emission operators. This is both true if either $B_+(x) \longrightarrow - B_+(x)^{\dagger}$ (trans-

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18). In the case of transformation I we could substitute

$$A_\mu(x) \longrightarrow -\hat{A}_\mu(x)$$

by the equivalent transformation

$$A_\mu(x) \longrightarrow A_\mu(x) \quad ; \quad e \longrightarrow -e$$ (change the sign of the charge.

formations I, IV) in which case positive particles go into positive particles, or:

$$B_+(x) \longrightarrow - B_+^*(x)^\dagger \quad \text{(transformations II, III)},$$

when positive particles go into negative particles.

Now we consider a general situation in which there are several spin 1/2 particles which interact with the electromagnetic field (via its charge or anomalous magnetic moment), a charged meson field (we take a pseudoscalar one, as an example) and an spinor field of zero mass.

The first group of spinor fields transform necessarily by I or II. These fields will be identified, in special examples to the proton, neutron, electron and μ-meson fields. Also it should be remembered that if one of them transforms, under time inversion, by I (or II) then all the others transform in the same way.

The Boson (pseudoscalar) field, whose transformation is conditioned by that of the charged fields in the way indicated in Table I, will be identified to the π-meson in some examples.

The zero mass field of spin 1/2 will be identified to the neutrino field, unless the opposite is specified. It may transform by any one of I, II, III or IV, under time inversion.

The interaction representation of Field Theory will be used from now on. The condition of relativistic invariance is now, besides that of invariance of the free field equations, the invariance of the interaction hamiltonian $\mathcal{H}(x)$ under the considered transformation.

We consider now the possible cases of combination of these transformations, the resulting covariant quantities and the restrictions which may result on the form of the interactions from the assumption of an special combination of transformations. We shall not go, however, into the systematic analysis of the cases when the neutrino field transforms by III or IV, as they will not bring any new possible form of Fermi interaction which will not be obtained in sec. D, by the introduction of phase factors in the improper transformations. On the contrary, they bring strong restrictions on the possible forms of interaction with the other fields. For instance, if the neutrino field $\psi_\nu(x)$ transforms by IV and the fields $\psi(x)$ by II, then no form of Fermi interaction is possible which is relativistically invariant. This because in view of the transformations:

$$\psi_\nu \longrightarrow \beta C \overline{\psi}_\nu^{\prime} = \beta \psi_\nu \; ; \; \psi \longrightarrow \beta \gamma_5 C \overline{\psi}^{\prime} = \beta \gamma_5 \psi \qquad \text{(II-32)}$$

we have, for time inversion:

$$\overline{\psi}\, \psi_\nu \longrightarrow \overline{\psi}\, \gamma_5\, \psi_\nu \qquad \text{(II-33)}$$

A similar situation happens when we take transformation III for the neutrino and I for the other fields. In the case when we use transformation III for the neutrino and II for the other fields then we are restricted, by the condition of invariance of $\mathcal{H}(x)$ under time inversion, to Fermi interactions of the type:

$$\mathcal{H}(x) = g\, \overline{\psi}_P(x)\, \psi_N(x)\, \overline{\psi}_e(x)\, (\psi_\nu(x) + \gamma_5 C\, \overline{\psi}_\nu(x)) + \text{h.c.} \qquad \text{(II-34)}$$

This because in this case ψ_ν transforms as:

$$\psi_\nu \longrightarrow \beta C \overline{\psi}_\nu = \beta \gamma_5 (\gamma_5 C \overline{\psi}_\nu) \qquad \text{(II-35)}$$

$$\gamma_5 C \overline{\psi}_\nu \longrightarrow \beta \gamma_5\, \psi_\nu \qquad \text{(II-36)}$$

Thus:

$$\psi_\nu + \gamma_5 C \overline{\psi}_\nu \longrightarrow \beta \gamma_5 (\gamma_5 C \overline{\psi}_\nu + \psi_\nu) \qquad \text{(II-37)}$$

In other terms the field quantity $\psi_\nu + \gamma_5 C \bar{\psi}_\nu$ transforms under time inversion of type III in the same way as the other fields ψ transform under the time inversion of type II:

$$\psi \longrightarrow \beta \gamma_5 \psi \qquad \text{(II-38)}$$

The possibility of an interaction of the form (II-34) will be found again in sec. D, when appropriate phase factors will be introduced in the transformations under the improper Lorentz group for the several fields involved in it.

We now consider the case when only transformations I or II are used, for the neutrino inclusive.

1) Covariant one particle quantities.

The covariant quantities formed with a given field $\psi(x)$ (and $\psi^\dagger(x)$) assume different form and covariance properties according to the type of transformation used for time invertion:

a). <u>Transformation I:</u>

$$\psi \longrightarrow \beta \gamma_5 C \bar{\psi}$$

In this case we have the usual covariant quantities:

Scalar: $\bar{\psi}\psi \longrightarrow \bar{\psi}\psi$

Vector: $\bar{\psi}\gamma_\mu\psi \longrightarrow -\bar{\psi}\hat{\gamma}_\mu\psi$

Tensor: $\bar{\psi}\gamma_\mu\gamma_\lambda\psi \longrightarrow -\bar{\psi}\hat{\gamma}_\mu\hat{\gamma}_\lambda\psi \qquad (\mu \neq \lambda)$

Pseudovector: $\bar{\psi}\gamma_5\gamma_\mu\psi \longrightarrow \bar{\psi}\hat{\gamma}_5\hat{\gamma}_\mu\psi$

Pseudoscalar: $\bar{\psi}\gamma_5\psi \longrightarrow \bar{\psi}\gamma_5\psi$

where: $\hat{\gamma}_0 = -\gamma_0; \ \hat{\gamma}_i = \gamma_i \ (i = 1,2,3) \ ; \ \hat{\gamma}_5 = -\gamma_5$

Thus we see from the transformation of the above quantities under time inversion that the scalar, pseudovector and pseudoscalar quantities have the ordinary variance under time inversion. The vector and tensor quantities transform with opposite sign (under this type of time inversion) in relation to the ordinary vector and tensor quantities. This was the reason why we had to assume similar transformation for the electro-magnetic field in the case when it interacts with a field transforming by I. Otherwise we would not have invariance of the equations of motion.

b). Transformation II.

$$\psi \longrightarrow \beta\gamma_5 C\,\overline{\psi'} = \beta\gamma_5\psi$$

In this case we find for the covariant quantities the expressions given by Case[15], all of them transforming in the ordinary way:

Scalar: $\overline{\psi}\,\psi + \overline{\psi'}\,\psi'$

Vector: $\overline{\psi}\,\gamma^{\mu}\psi - \overline{\psi'}\,\gamma^{\mu}\psi'$

Tensor: $\overline{\psi}\,\gamma^{\mu}\gamma^{\lambda}\psi - \overline{\psi'}\,\gamma^{\mu}\gamma^{\lambda}\psi'$ $\quad (\mu \neq \lambda)$

Pseudovector: $\overline{\psi}\,\gamma^{5}\gamma^{\mu}\psi + \overline{\psi'}\,\gamma^{5}\gamma^{\mu}\psi'$

Pseudoscalar: $\overline{\psi}\,\gamma^{5}\psi + \psi'\,\gamma^{5}\psi'$

Thus we see that Case's conclusion that condition of relativistic invariance of a field theory under time inversion imposes the form given above for the covariant quantities formed with the field ψ is only correct for the type II of transformation under time inversion.

2) Two-fields covariant quantities.

When we consider the covariant quantities formed with two different spinor fields we should allow the possibility that they transform in different ways under time inversion. We have then the following cases:

a) Both fields transform under time inversion by transformation I

In this case we find, for instance, the transformation:

$$\overline{\Psi}\,\Psi_\nu \longrightarrow \overline{\Psi}_\nu\,\Psi \qquad \text{(II-39)}$$

for the quantity $\overline{\Psi}\,\Psi_\nu$, usually considered as an scalar. Similar transformations are found for the other quantities usually considered as vector, tensor, etc. Thus we see that in this case the scalar quantity would be the following one:

$$\overline{\Psi}\,\Psi_\nu + \overline{\Psi}_\nu\,\Psi \qquad \text{(II-39)}$$

However, we will find in the following analysis that, although $\overline{\Psi}\,\Psi_\nu$ is not invariant under time inversion the usual scalar meson interaction and scalar Fermi interaction are still invariant under time inversion.

b) Both fields transform by II.

Here we find again the usual covariant quantities:

Scalar: $\overline{\Psi}\,\Psi_\nu \longrightarrow -\Psi_\nu\,\overline{\Psi} = \overline{\Psi}\,\Psi_\nu$

Vector: $\overline{\Psi}\,\gamma^\mu\,\Psi_\nu \rightarrow -\Psi_\nu\,\hat{\gamma}^\mu_T\,\overline{\Psi} = \overline{\Psi}\,\hat{\gamma}^\mu\,\Psi_\nu$

Tensor: $\overline{\Psi}\,\gamma^\mu\gamma^\lambda\,\Psi_\nu \rightarrow -\Psi_\nu\,\hat{\gamma}^\lambda_T\,\hat{\gamma}^\mu_T\,\overline{\Psi} = \overline{\Psi}\,\hat{\gamma}^\mu\hat{\gamma}^\lambda\,\Psi_\nu \qquad (\mu \neq \lambda)$

Dual Tensor: $\overline{\Psi}\gamma^5\gamma^\mu\gamma^\lambda\,\Psi_\nu \rightarrow -\Psi_\nu\,\hat{\gamma}^\lambda_T\,\hat{\gamma}^\mu_T\,\hat{\gamma}^5_T\,\overline{\Psi} = \overline{\Psi}\,\hat{\gamma}^5\hat{\gamma}^\mu\hat{\gamma}^\lambda\,\Psi_\nu \qquad (\mu \neq \lambda)$

Pseudovector: $\overline{\Psi}\,\gamma^5\gamma^\mu\,\Psi_\nu \rightarrow -\Psi_\nu\,\hat{\gamma}^\mu_T\,\hat{\gamma}^5_T\,\overline{\Psi} = \overline{\Psi}\,\hat{\gamma}^5\hat{\gamma}^\mu\,\Psi_\nu$

Pseudoscalar: $\overline{\Psi}\,\gamma^5\,\Psi_\nu \rightarrow -\Psi_\nu\,\hat{\gamma}^5_T\,\overline{\Psi} = \overline{\Psi}\,\hat{\gamma}^5\,\Psi_\nu$

The transformation properties of these quantities under time inversion (transformation II) is also indicated. We see that they have the ordinary covariance only if the two different fields anticommute. This anticommutation of different fields, which is also necessary when we pass from the interaction representation to the Heisenberg representation (see Part I), was used in the last step of these transformations.

It should be observed here that if these fields transform both by II, but with opposite signs (the possibility of the introduction of phase factors ± 1 or $\pm i$ for the improper transformations will be analysed in sec. D), then the transformation properties of the expressions listed above will be exchanged as follows:

$$\begin{aligned} &\text{Scalar} \longleftrightarrow \text{Pseudoscalar} \\ &\text{Vector} \longleftrightarrow \text{Pseudovector} \\ &\text{Tensor} \longleftrightarrow \text{Dual tensor} \end{aligned}$$

c). If ψ_ν transforms, under time inversion, by I and ψ by II then we find that the quantities listed in sec. b) are not anymore covariant, as under time inversion we have then:

$$\bar{\psi}\,\psi_\nu \longrightarrow \bar{\psi}\;C\,\bar{\psi_\nu} \qquad \text{(II-40)}$$

However, as we also find:

$$\bar{\psi}\,C\,\bar{\psi_\nu} \longrightarrow \bar{\psi}\;\psi_\nu \qquad \text{(II-41)}$$

we see that $\bar{\psi}\,(\psi_\nu + C\,\bar{\psi_\nu})$ is an scalar. In general, the covariant quantities are those obtained from the corresponding ones listed in sec. b) by the substitution:

$$\psi_\nu \longrightarrow \psi_\nu + C\,\bar{\psi_\nu} \qquad \text{(II-42)}$$

d). If ψ_ν transforms by II and ψ by I the covariant quantities will be of the form:

$$\overline{\psi_\nu}\ (\psi + C\,\overline{\psi}), \qquad \text{(II-43)}$$

say, the scalar one.

(3) Meson interactions.

Here we have two possible cases

a) Both spinor fields transform in the same way.

In this case we find for instance for the pseudoscalar interaction with pseudoscalar charged meson the usual expression:

$$\mathcal{H}(x) = g(B(x)\,\overline{\psi}_1(x)\,\gamma^5\,\psi_2(x) + \text{h.c.}) \qquad \text{(II-44)}$$

It should be observed that in the case of transformation I (see Table I for the transformation of B(x)) each term in (II-44) go into the other one when the time inversion is performed. In the case of transformation II each term is invariant by itself.

b) The two fields transform, one by I and the other by II (the transformation of B(x) is determined by that of the charged field according to the Table I).

Here we find, instead of (II-44) the invariant interaction:

$$\mathcal{H}(x) = g\left[B(x)\,\overline{\psi}_1(x)\,\gamma^5(\psi_2(x) + C\,\overline{\psi_2}(x)) + \text{h.c.}\right] \qquad \text{(II-45)}$$

It is clear that this type of interaction (II-45) cannot be used for the interaction of the π meson with the nucleons as it would lead to instability of the atomic nuclei[19]. However, it could be used, as well as (II-44) for the description of the π-decay:

$$\pi \longrightarrow \mu + \nu \qquad \text{(II-46)}$$

if ψ_1 is taken as the μ meson field and ψ_2 as the neutrino field.

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19). For instance, if ψ_1 is the proton field and ψ_2 the neutron field then the interaction (II-45) would lead to the second order process in which a proton would transformin an antineutron with emission of a π^+ meson and then the antineutron would be anihilated with another proton with emission of a π^- meson.

(4) Fermi interactions of β-decay.

Again as in sec. 3) we have two different cases.

a) All spinor fields involved in the interaction transform in the same way.

We find in this case the usual interaction, say for the scalar theory:

$$\mathcal{H}(x) = g\left[\overline{\Psi}_P(x)\,\Psi_N(x)\;\;\overline{\Psi}_e(x)\;\;\Psi_\nu(x) + \text{h.c.}\;\right] \qquad \text{(II-45)}$$

b) The neutrino field transforms in different way then the other fields.

As we are considering only the transformations I and II for the neutrino as for the other particles we find that the possible interaction is of the form:

$$\mathcal{H}(x) = g\left[\overline{\Psi}_P(x)\,\Psi_N(x)\;\;\overline{\Psi}_e(x)\;(\,\Psi_\nu(x) + C\,\overline{\Psi}_\nu(x)\;) + \text{h.c.}\right] \qquad \text{(II-46)}$$

in the two possible situations which can occur.

An interaction of the type (II-46) will also be found as possible in the following section. This possibility will result there from the use of appropriate phase factors, as will be introduced there, in the transformations of the impróper Lorentz group.

D. Phase factors in the improper Lorentz transformations of spinor fields.

The possibility of introducing a phase factor ± 1 or $\pm i$ in the transformations, under the improper Lorentz group, of a spinor field Ψ :

$$\Psi \longrightarrow \Lambda\,\Psi \qquad \text{(II-47a)}$$

was first shown by Racah[20]. He argued that Pauli's condition[21],

$$\det\Lambda = 1 \qquad \text{(II-47b)}$$

would still be satisfied if we substitute Λ by $f\,\Lambda$ in (II-47a), if the

20). G. Racah, Nuovo Cim., 14, 322, 1937.

21). W.Pauli, Ann. Inst. H. Poincare', 6, 109, 1936.

phase factor f is equal to ± 1 or $\pm i$. Such a phase factor could not be introduced for the continuous restricted transformations as the limiting value of Λ for no change of coordinates should be the unit matrix. However, Racah introduced a symmetry principle[20], whose physical meaning is not clear to us, by which the antiparticle field ($\psi' = C\bar{\psi}^*$) should transform in the same way as the particle field (ψ).

Recently this question of phase factors in the improper transformations was reanalyzed by Yang and the present author[22], Racah's principle discarded, and inferences were drawn for the interaction of several fields.

This analysis will be reported, in more detail, in the present section for the sake of future application in the following Parts of this work.

1) Phase factors and conservation of particles.

Besides Racah's justification of the introduction of the phase factors in the improper transformations of spinor fields there is another argument which is more appropriate as a justification of this possibility for the case of antiunitary time inversions. It is known that the proper transformations for spinor fields are double valued as a consequence of the fact that for an space rotation of 360 degrees, which is physically equivalent to no rotation at all, we have the transformation:

$$\psi \longrightarrow -\psi \qquad \text{(II-48a)}$$

in contrast to the case of no rotation when we have

$$\psi \longrightarrow \psi \qquad \text{(II-48b)}$$

(We use, as before, Heisenberg type of transformation).

In other words, two consecutive rotations of 180°, which are physically equivalent to no rotation at all, bring about a change of sign

22). C. N. Yang and J. Tiomno, Phys. Rev. 79, 495, 1950.

in the field operators $\Psi(x)$. Thus there is no reason why we should exclude the similar possibility of a change of sign of $\Psi(x)$ after two improper transformations[23], which are physically equivalent to no transformation at all.

Thus we see that, if we take for the improper transformations[24],

$$\Psi \longrightarrow f\ \beta\ \Psi \qquad \text{(space reflexion)} \qquad \text{(II-49)}$$

$$\Psi \longrightarrow i\ f_I \beta \gamma_5 C\ \overline{\Psi} \qquad \text{(time inversion I)} \qquad \text{(II-50a)}$$

$$\Psi \longrightarrow i\ f_{II} \beta \gamma_5 C\ \overline{\Psi}' \qquad \text{(time inversion II)} \qquad \text{(II-50b)}$$

(where f, f_I and f_{II} are not necessarily the same for different fields) We find that, after two identical transformations of one of these forms the field operator $\Psi(x)$ will change as:

$$\Psi(x) \longrightarrow f^2\, \Psi(x) \quad \text{or}\quad \Psi(x) \longrightarrow f_{II}^2\, \Psi(x) \qquad \text{(II-51a)}$$

in the cases of (II-49) and (II-50b) respectively, and:

$$\Psi(x) \longrightarrow -f_I^*\ f_I\, \Psi(x) \qquad \text{(II-51b)}$$

in the case of (II-50a).

Thus we see that for (II-49) and (II-50b) $\Psi(x)$ will be unchanged if:

$$f = \pm 1 \quad \text{or}\ f_{II} = \pm 1\,, \qquad \text{(II-52a)}$$

respectively, and will change sign if:

$$f = \pm i \quad \text{or}\ f_{II} = \pm i, \qquad \text{(II-52b)}$$

respectively. In the case of (II-50a) $\Psi(x)$ will change sign, regardless

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23). It is of immediate verification that if $\Psi(x)$ changes sign after two such transformations, the wave functions $\varphi_\alpha(x)$, $\varphi_{\alpha_1\alpha_2}(x_1x_2)$, etc., in configuration representation change sign for the sub-spaces of odd number of particles and is unaltered for those of even number of such particles.

24). It should be kept in mind that for time inversion the additional operation $F_1(x)\,F_2(x) \longrightarrow \tilde{F}_2(\tilde{x})\,\tilde{F}_1(\tilde{x})$ should be performed together with (II-50) of the value of f_I, which should, however, be unimodular for the conservation of normalization:

$$f^{*}_{I} \; f_{I} = 1 \qquad \text{(II-53)}$$

We did not consider the cases of transformations III and IV because in view of the argument given in the preceding section, they are of no interest for our future analysis.

From now on we shall arbitrarily assume, transformation II, i.e. (II-50b) for the charged particles (and neutrons), in view of the fact that in this case the electromagnetic potential $A_{\mu}(x)$ transforms, under time inversion, as an ordinary covariant vector (see Table I). This is not, however, a restriction of generality as, it is easy to see, the assumption of transformation I , i.e., (II-50a) for these particles would not change the qualitative results of the following analysis.

Transformation I will be retained, as a possibility, for the neutrino case. As a consequence of the condition for invariance of the interaction hamiltonian $\mathcal{H}(x)$ we will be then restricted to the values ± 1, $\pm i$ for f_I.

Finally, in order not to introduce scalar-like quantities which would be invariant under space reflexion (time inversion) and change sign under time inversion (space reflexion) we take from now on (also arbitrarily in principle):

$$f_I = \pm f \quad ; \quad f_{II} = \pm f \qquad \text{(II-54)}$$

It should be observed here that the usual assumption that a phase factor in the field operators $\psi^{(\rho)}(x)$ is irrelevant applies only to the cases when there is <u>conservation of particles</u>, i.e., when in a given process a certain particle disappears either another particle could be anihilated, instead, or an antiparticle (not a particle) could be created. This because the mathematical expression of such a conservation principle is just that $\mathcal{H}(x)$ should be invariant under a transformation by a phase factor η, the same for all particles and η^* for the antiparticles. Now an interaction of the type:

$$\mathcal{H}(x) = g\,\overline{\psi}_P(x)\,\psi_N(x)\;\overline{\psi}_e(x)\left[\psi_\nu(x) + c\,\overline{\psi}_\nu(x)\right] + \text{h.c.} \qquad \text{(II-55)}$$

which is invariant[25] under, say, (II-49) and (II-50b) with $f = \pm i$ ($f_{II} = f$) for all particles, but not with $f = \pm 1$, is surely not invariant under a phase transformation of the type above referred to (say, for instance with $\eta = i$). The physical meaning of this fact is that interaction (II-55) leads to no conservation of particles as either a neutrino or antineutrino can be emitted, in this case, in a given process. However, interaction (II-55) still satisfies the conditions of conservation of charge and conservation of nucleons, mathematically expressed by its invariance under an arbitrary phase transformation for the charged particles or for the nucleons, respectively. This example should be enough to show the importance of the phase factors f in the improper transformations.

2) Phase factors in the improper transformations for nucleons and the symmetry properties of the π meson.

As it was shown by Yang and Tiomno[22] the fact that we do not know the relative signs of the improper transformation for the proton and

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25). This will be shown in section 4).

neutron field would leave undetermined the reflexion properties of the π meson field even if the form of the interaction of these particles would be experimentally found. As an example, if we assume an interaction of the type:

$$\Lambda_\pi \overline{\Psi}_P \gamma^5 \Psi_N \; + \; \text{h.c.} \qquad \text{(II-56)}$$

thus the π meson would be pseudoscalar, if Ψ_P and Ψ_N transform in the same way under the improper Lorentz group, or scalar, if they transform with opposite signs. This because $\overline{\Psi}_P \gamma^5 \Psi_N$ which is a pseudoscalar quantity in the first case, is an scalar in the second case.

3) Signs of the improper Lorentz transformations and the representations of the Dirac equations for a system of two or more interacting fields:

It is interesting to relate the results of the previous section to the fact that when there are two interacting spinor fields there are two different representations for the Dirac equations of these fields (we consider the interaction representation): that in which the mass terms in the equations for the two fields:

$$(\gamma^\mu \frac{\partial}{\partial x^\mu} \pm M_{(r)}) \Psi^{(r)}(x) = 0 \quad (r = 1,2) \qquad \text{(II-57)}$$

have the same sign and that in which they have opposite signs.

Now we see that if we start from the case when $\Psi^{(1)}$ and $\Psi^{(2)}$ do satisfy Dirac equations with the same sign of the mass term, but transform under the improper group (say by I) with opposite signs and make a transformation:

$$\Psi^{(1)} \rightarrow \Psi^{(1)} \quad , \Psi^{(2)} \rightarrow \gamma_5 \Psi^{(2)} \qquad \text{(II-57)}$$

the new fields will transform now in the same way but will satisfy Dirac equations with opposite signs of the mass term. Also the quantity

$\overline{\Psi}(1)\,\gamma_5\,\psi(2)$, which was a scalar in the first representation will go into $\overline{\Psi}\,(1)\;\psi\,(2)$, a scalar in the new representation.

Another interesting example is the comparison of the Fermi theories characterised by the interactions:

$$\mathcal{H}_1(x) = g\,\overline{\Psi}_P(x)\;\psi_N(x)\;\overline{\Psi}_e(x)\;\psi_\nu(x) + \text{h.c.} \qquad \text{(II-58)}$$

$$\mathcal{H}_2(x) = g\,\overline{\Psi}_P(x)\;\psi_N(x)\;\overline{\Psi}_e(x)\,\gamma^5\;\psi_\nu(x) + \text{h.c.} \qquad \text{(II-59)}$$

when the same sign of the mass term is taken for all particles. $\mathcal{H}_1(x)$ is a scalar density if, say, all particles transform by I with the same phase factor. $\mathcal{H}_2(x)$ is an invariant if in the improper transformation for the neutrino the opposite sign is used in relation to those for the other fields. Now if we make the transformation:

$$\psi_\nu \longrightarrow \gamma_5\;\psi_\nu \qquad \text{(II-60)}$$

then $\mathcal{H}_2(x)$ will go into $\mathcal{H}_1(x)$ but the mass term of the Dirac equation for the neutrino will change sign. Thus we see that the theories characterized by $\mathcal{H}_1(x)$ and $\mathcal{H}_2(x)$ will be equivalent if the mass of the neutrino is zero.

4) Transformation properties of the fields:

$$\Psi' = C\,\overline{\Psi}\;;\;\Psi'' = \gamma_5\,C\;\;\overline{\Psi} \qquad \text{(II-61)}$$

These quantities are known to be the only ones formed with $\psi^\dagger(x)$ which transform as $\psi(x)$ under the proper Lorentz group. Their transformations under the improper group are important in order to decide about the invariance of theories in which they appear in the interaction hamiltonian in linear combination with $\psi(x)$ (say, for the neutrino field) as we shall consider in Parts III and IV.

Now, if we consider the improper transformations (II-49), (II-50), with the condition (II-54) we see that[20), 23)]:

a) Ψ' transforms in the same way as Ψ (for example, $\Psi' \longrightarrow f\beta\Psi'$ for space reflexion) if

$$f = \pm i \qquad \text{(II-62a)}$$

and with opposite sign (say, $\Psi' \longrightarrow -f\beta\Psi'$) if:

$$f = \pm 1 \qquad \text{(II-63a)}$$

b) Ψ'' transforms as Ψ if

$$f = \pm i \qquad \text{(II-62b)}$$

and with opposite sign if

$$f = \pm i \qquad \text{(II-63b)}$$

These results lead to the conclusion (say, for time inversion I) that the Fermi interaction:

$$\mathcal{H}_1(x) = g\,\overline{\Psi_P}(x)\,\Psi_N(x)\,\overline{\Psi_e}(x)\left[\Psi_\nu(x) + \lambda\, C\,\overline{\Psi_\nu}(x)\right] + \text{h.c.} \qquad \text{(II-64)}$$

where λ is an arbitrary constant ; is invariant if:

$$f = \pm i \qquad \text{(II-62a)}$$

(we assume, for simplicity, the same f for all particles involved). In this case the interaction

$$\mathcal{H}_2(x) = g\,\overline{\Psi_P}(x)\,\Psi_N(x)\,\overline{\Psi_e}(x)\left[\Psi_\nu(x) + \lambda\gamma_5 C\,\overline{\Psi_\nu}(x)\right] + \text{h.c.} \qquad \text{(II-65)}$$

will not be invariant as the term in λ will change sign under the improper transformations.

If, however, we have, instead of (II-62a):

$$f = \pm 1 \qquad \text{(II-62b)}$$

then the situation will be reversed, $\mathcal{H}_2(x)$ being invariant and $\mathcal{H}_1(x)$ not invariant.

Two other possibilities:

$$\mathcal{H}_3(x) = g\, \overline{\psi}_P(x)\, \gamma_5\, \psi_N(x)\, \overline{\psi}_e(x) \left[\psi_\nu(x) + \lambda C\, \overline{\psi}_\nu(x)\right] + \text{h.c.} \qquad \text{(II-66)}$$

and

$$\mathcal{H}_4(x) = g\, \overline{\psi}_P(x)\, \gamma_5\, \psi_N(x)\, \overline{\psi}_e(x) \left[\psi_\nu(x) + \lambda \gamma_5\, C\, \overline{\psi}_\nu(x)\right] + \text{h.c.} \qquad \text{(II-67)}$$

will arise if in the transformations (II-62a) and (II-62b), respectively, we take the opposite sign, say for the neutron, of that used for the other particles.

The consideration of the most general case (different phase factors for the several particles and different types of time inversion for the neutrino and the other particles) will not give rise to any new situation not included in $\mathcal{H}_1(x)$, $\mathcal{H}_2(x)$, $\mathcal{H}_3(x)$ and $\mathcal{H}_4(x)$.

4) Transformations of a Majorana field.

A Majorana field[26), 27)] $\mathcal{U}(x)$ is characterized by the self charge conjugation property:

$$\mathcal{U}(x) = C\, \overline{\mathcal{U}}(x) = \mathcal{U}'(x) \qquad \text{(II-68)}$$

This condition is the analogous of the hermiticity condition for neutral Boson fields. Indeed (II-68) takes the form

$$\mathcal{U}(x) = \mathcal{U}^\dagger(x) \qquad \text{(II-68a)}$$

in the representation of the γ^μ matrices used by Majorana[26)]

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26). E. Majorana, Nuovo Cim., 14, 171, 1937.

27). Furry, Phys. Rev., 54, 56, 1938.

Such a type of field can be used only to describe neutral particles with no magnetic moment[26,27]. The possibility of its use for the description of neutrinos was first suggested by Majorana[28] and will be analysed in the following Parts of this work.

The transformation of $\mathcal{U}$ (x) under the proper Lorentz group is the same as for a field of the type ψ (x) (which we shall denote from now on as a Dirac field).

In what concerns to the transformation under the improper group we should observe that:

1) Transformations I and II for time inversion become identical:

$$\mathcal{U} \longrightarrow i\, f \beta \gamma_5\, \mathcal{U} = i\, f \beta \gamma_5\, C\, \overline{\mathcal{U}} \qquad \text{(II-69)}$$

2) Transformations III and IV (see Table 1), which become identical for a Majorana field:

$$\mathcal{U} \longrightarrow \beta\, \mathcal{U} = \beta\, C\, \overline{\mathcal{U}} \qquad \text{(II-68)}$$

is excluded for the neutrino, which is known to interact with charged fields. This is so because if the electron field, for instance, transforms as:

$$\psi_e \longrightarrow \beta \gamma_5\, C\, \overline{\psi_e'} = \beta \gamma_5 \psi_e \qquad \text{(II-68a)}$$

then the quantity $\overline{\psi_e}\, \mathcal{U}$ transforms as:

$$\overline{\psi_e}\, \mathcal{U} \longrightarrow -\overline{\psi_e} \gamma_5\, \mathcal{U} \qquad \text{(II-70)}$$

under time inversion. As a consequence of (II-70) no possible Fermi interaction would exist, which would be invariant under the improper Lorentz group.

3) For space reflexion which now is given by:

$$\mathcal{U} \longrightarrow f\beta\,\mathcal{U} \qquad \text{(II-71)}$$

and the only remaining transformation (II-69) for time inversion, the only possible values for the phase factor f are:

$$f = \pm i \; . \qquad \text{(II-62a)}$$

These are the only values of f consistent with the transformations of the vanishing quantity $\mathcal{U} - C\overline{\mathcal{U}}$:

$$O = \mathcal{U} - C\overline{\mathcal{U}} \longrightarrow \beta\,(f\,\mathcal{U} + f^{*}\,C\,\overline{\mathcal{U}}), \qquad \text{(II-72)}$$

$$O = \mathcal{U} - C\overline{\mathcal{U}} \longrightarrow i\beta\gamma_5\;(f\,\mathcal{U} + f^{*}\,C\,\overline{\mathcal{U}}), \qquad \text{(II-73)}$$

for space reflexion and time inversion, respectively.

Now the invariant interaction hamiltonians of the Fermi type are of the form:

$$\mathcal{H}(x) = g\,\overline{\psi}_P(x)\,\psi_N(x)\,\overline{\psi}_e(x)\,\mathcal{U}(x) + h.c. \qquad \text{(II-75)}$$

if, for instance, the Dirac fields transform according to (II-49) and (II-50b) and the neutrino field $\mathcal{U}(x)$ according to (II-69) and (II-71) and

$$f = \pm i \; , \qquad \text{(II-62a)}$$

the same for all particles. They will be of the form:

$$\mathcal{H}'(x) = g\,\overline{\psi}_P(x)\,\gamma_5\,\psi_N(x)\,\overline{\psi}_e(x)\,\mathcal{U}(x) + h.c. \qquad \text{(II-75)}$$

if for one of the fields we take the opposite sign in (II-62a). Other possible assignments of the values of f for the several fields will lead to (II-74) or to (II-75).

PART III

STRUCTURE OF THE UNDERLYING HILBERT SPACE OF THE SPINOR FIELD OPERATORS

A) General considerations.

In the field theories (we consider only spinor fields here) we deal with field operators $\psi^{(\ell)}(x)$ corresponding to the several types of particles involved in the theory. If we consider, for instance, the interaction representation and forget, for the moment, the interaction, i.e., if we put:

$$\frac{\delta \Psi[\sigma]}{\delta \sigma(x)} = 0 \qquad \text{(III-1)}$$

then a relatively general solution of (III-1) can be written as:

$$\Psi[\sigma] = \int_{\sigma} d\sigma_{\mu} \; \bar{\psi}_{+}(x) \; \gamma^{\mu} \varphi(x) \, \Omega_{0} \qquad \text{(III-2)}$$

for a one particle state, $\psi(x)$ being one of the spinor fields in the theory. In (III-2) $\psi_{+}(x)$ is the positive energy part[2] of $\psi(x)$ and $\varphi(x)$ is the wave function in configuration representation of our one particle system. Ω_0 is the vacuum wave function characterized by:

$$\psi_{+}(x)\Omega_0 = 0 \qquad \text{(III-2a)}$$

The fact that $\psi_{+}(x)$ satisfies the Dirac equation:

$$(\gamma^{\mu} \frac{\partial}{\partial x^{\mu}} + m)\psi_{+}(x) = 0, \qquad \text{(III-3)}$$

together with (III-1,2), leads to the condition:

$$(\gamma^{\mu} \frac{\partial}{\partial x^{\mu}} + m) \varphi(x) = 0 \qquad \text{(III-4)}$$

for the wave function $\varphi(x)$.

A complete set of (positive energy) wave functions $\varphi(x)$ is, for instance, that of the plane waves associated to all possible values of the momentum $\vec{k}$ and the two values of the spin coordinate.

However, besides $\Psi_+(x)$, also the negative energy part $\Psi_-(x)$ of $\Psi(x)$ appears in general in the theory. From the condition of relativistic invariance of the theory we cannot, except for a few special cases, discard the negative energy part of the operator $\Psi(x)$ (see part I). We have, thus, to give an appropriate meaning to its role in the theory. The usual procedure is to introduce a new set of wave functions, analogous to $\varphi(x)$, describing another type of particle associated to the same field $\Psi(x)$. These are the antiparticle wave functions $\varphi'(x)$. Thus we are led to a more general expression for the one particle wave function Ψ:

$$\Psi[\sigma] = \int_\sigma d\sigma_\mu \left[\overline{\Psi}_+(x)\,\gamma^\mu\,\varphi(x) + \overline{\Psi}{}'_+(x)\,\gamma^\mu\,\varphi'(x)\,\Omega_0 \right. \qquad \text{(III-5)}$$

where $\Psi'_+(x)$ is another positive energy field (we shall use the expression "half field" [28] to denote such positive energy fields as $\Psi_+(x)$, $\Psi'_+(x)$) which latter is to be related to $\Psi_-(x)$. The usual relation is:

$$\Psi_-(x) = C\,\overline{\Psi}{}'_+(x), \qquad \text{(III-6)}$$

or $\Psi'_+(x)$ is the positive energy part of the charge conjugate field $\Psi'(x)$

$$\Psi'(x) = C\,\overline{\Psi}(x). \qquad \text{(III-7)}$$

Also we have

$$\Psi'_-(x) = C\,\overline{\Psi}_+(x) \qquad \text{(III-6a)}$$

and the condition:

$$\Psi'_+(x)\,\Omega_0 = 0 \qquad \text{(III-8)}$$

28). See, for instance, reference 4).

In general we can proceed in the reverse way. We decompose $\psi(x)$, in a relativistic way, into two parts:

$$\psi(x) = \psi_1(x) + \psi_2(x) \qquad \text{(III-9)}$$

and express both $\psi_1(x)$ and $\psi_2(x)$ in terms of two half fields. Now, instead of the particle and antiparticle wave functions we will have two other types of wave functions $\varphi_1(x)$ and $\varphi_2(x)$ corresponding to the two kinds of particles associated with the field $\psi(x)$.

We shall call such fields $\psi(x)$ to which correspond two types of particles, in the sense referred to above, Dirac fields. The several possible ways in which the operation described above can be performed will lead in general to different representations of the theory as will be analysed in the following sections.

If, however, one of the spinor fields (which we denote by $\mathcal{U}(x)$) satisfies a condition such as:

$$\mathcal{U}(x) = C\,\overline{\mathcal{U}}(x) \qquad \text{(III-10)}$$

then an irreducible representation will be obtained if we express $\mathcal{U}(x)$ as:

$$\mathcal{U}(x) = \mathcal{U}_+(x) + C\,\overline{\mathcal{U}_+}(x) \qquad \text{(III-11)}$$

where $\mathcal{U}_+(x)$ is the positive energy part of $\mathcal{U}(x)$. It is clear that $\mathcal{U}(x)$ given by (III-11) satisfies the condition (III-10). Now only one set of wave functions $\varphi_M(x)$ is necessary to describe all the one particle states associated with this field.

We shall call the field $\mathcal{U}(x)$ a Majorana field[26),27)]. It will be seen that the irreducible representation of the Majorana field operators has a dimension which is the square root of that of a Dirac field operator.

Finally we should anticipate that for some special types of theory if we decompose a Dirac field $\psi(x)$, in an appropriate way, into two half fields $\Psi_1(x)$ and $\Psi_2(x)$, then we can express the interaction Hamiltonian in terms of only one of them. In other words, although in this case there are two kinds of particles associated with the field $\Psi(x)$, only one of them will be involved in the actual processes resulting from the interaction used. This will happen in the theories that we shall call "Projection Theories".

B. Invariant decomposition of a field $\Psi(x)$.

We want to find the possible relativistically invariant ways of decomposing a Dirac field $\Psi(x)$ into two parts $\Psi_1(x)$ and $\Psi_2(x)$. This is necessary, as shown in sec. A, in order to express $\Psi(x)$ in terms of two half fields (positive energy fields) and their hermitian conjugate. These two half fields will be interpreted as operators of anihilation of the two kinds of particles associated to the field $\Psi(x)$; both of them will give a vanishing result when applied to the vacuum wave function Ω_0.

The usual decomposition of $\Psi(x)$ into the positive and negative energy parts is one of such decompositions:

$$\Psi(x) = \Psi_+(x) + \Psi_-(x) \qquad \text{(III-12)}$$

where:

$$\Psi_+(x) = P_+\Psi(x) = i\int_\sigma d\sigma'_\mu\, S_+(x-x')\,\gamma^\mu\,\psi(x') \qquad \text{(III-13a)}$$

$$\Psi_-(x) = P_-\Psi(x) = i\int_\sigma d\sigma'_\mu\, S_-(x-x')\,\gamma^\mu\psi(x') \qquad \text{(III-13b)}$$

In (III-13), $S_+(x-x')$ and $S_-(x-x')$ are respectively, the positive and the negative energy parts of $S(x-x')$[2].

The operators P_+ and P_- defined in (III-13a) are operators of projection, as they satisfy the conditions usually imposed for two linear projection operators P_1 and P_2, defined by:

$$\psi(x) = \psi_{(1)}(x) + \psi_{(2)}(x) \qquad \text{(III-14)}$$

$$\psi_r(x) = P_r \psi(x), \quad (r = 1,2) \qquad \text{(III-14a)}$$

$$P_1 + P_2 = 1 \qquad \text{(III-15a)}$$

$$P_r P_s = \delta_{rs} P_r, \quad (r,s = 1,2) \qquad \text{(III-15b)}$$

$$P_r(\psi^1 + \psi^2) = P_r \psi^1 + P_r \psi^2 \qquad \text{(III-15c)}$$

$$P_r(\alpha \psi) = \alpha P_r \psi, \quad (r = 1,2) \qquad \text{(III-15d)}$$

(in (III-15d) α is a number).

Also, the decomposition (III-12) is relativistically invariant as, for any homogeneous Lorentz transformation $\mathcal{L}$ we find that:

$$\mathcal{L} P_r \psi = P_r \mathcal{L} \psi \quad (r = 1,2) \qquad \text{(III-16)}$$

where P_1 and P_2 represent, in this example, P_+ and P_-, respectively.

Finally if the decomposition (III-14) is made at a certain time it should be maintained for all future times (in interaction representation, say). This will be true if we have:

$$\left(\gamma^\mu \frac{\partial}{\partial x^\mu} + m\right) P_r \psi(x) = 0 \qquad \text{(III-17a)}$$

This last condition is equivalent to saying that the decomposition of $\psi(x)$ involves only the values of this field in a given spacelike surface which contains the point x:

$$P_r \psi(x) = i \int_\sigma S(x-x') \gamma^\mu P_r \psi(x') \, d\sigma'_\mu \qquad \text{(III-17b)}$$

This is true for P_+ and P_- defined by (III-13).

Other possible operators of projection, besides P_+ and P_- given above, may be used in order to decompose the field $\psi(x)$ in a relativistically invariant way. Such operators should satisfy the conditions (III-15), (III-16) and (III-17). This question will be analysed in sub-section 1).

It is not necessary, however, that the operators used in the decomposition of the field should be projection operators. However, the conditions of relativistic invariance, (III-16) and (III-17) should still be imposed on such a decomposition. A possible decomposition of this type will be considered in sub-sec. 2).

In subsection 4) these results will be applied in order to obtain the possible ways of expressing $\psi(x)$ in terms of two half-fields (and their hermitian conjugates). These possibilities will be used in sec. C, in the analysis of the structure of the underlying Hilbert space of the field theories.

1) Projection operators.

For the sake of generality it is convenient to allow a slight departure from the conditions (III-15), (III-16) and (III-17) imposed on the invariant projection operators.

First the homogeneity condition (III-15d) will be imposed only for α real. This is done in order that operations of the type introduced by Furry[27] will be still called "projection operations" (as Furry's does); these projection operations involve, besides $\psi(x)$ the hermitian conjugate $\psi^\dagger(x)$.

Also the invariance condition (III-16) will be allowed to take the form:

$$\mathcal{L}_i P_1 \psi = \pm P_2 \mathcal{L}_i \psi; \quad \mathcal{L}_i P_2 \psi = \pm P_1 \mathcal{L}_i \psi. \qquad \text{(III-16a)}$$

for improper transformations.

This last condition (III-16a) is still satisfactory in expressing the invariance of the decomposition, although the roles of P_1 and P_2 are exchanged under the improper transformation $\mathcal{L}_1$.

Now we consider the possible types of projection operators. These can be divided into two classes:

a) Projection operators not involving hermitian conjugation.

b) Projection operators involving hermitian conjugation.

In the case a) we find that we are restricted to the case of the operators P_+ and P_- defined by (III-12) and (III-14), in view of the invariance conditions (III-16) and (III-17).

We shall call these operators, Schrödinger projection operators, as they are equivalent to the differential operators proposed by Schrödinger[29] for the decomposition of a spinor wave function into its positive and negative energy parts:

$$P_+ \psi(x) = \frac{E + H}{2E} \psi(x) \qquad \text{(III-18a)}$$

$$P_- \psi(x) = \frac{E - H}{2E} \psi(x) \qquad \text{(III-18b)}$$

where:

$$E = +\sqrt{m^2 - \Delta} \; ; \; H = -i\vec{\alpha}\cdot\vec{\nabla} - m\beta \qquad \text{(III-18c)}$$

In case b), in view of the condition (III-16) of invariance of the projection under the Lorentz group we are restricted to the linear combinations of ψ with $C\,\overline{\psi}$ or $\gamma_5 C\,\overline{\psi}$ as these last two quantities are the only ones, formed with ψ^+, which transform in the same way as ψ under the restricted homogeneous transformations. In the subsection 2) we will show that projection operations cannot be formed which combine ψ with $\gamma_5 C\overline{\psi}$. Thus the only possible projection operators in this class are P_1 and P_2 given

29). E. Schrödinger, Berl. Ber., p. 418, 1930.

by:

$$P_1 \Psi = \frac{1}{2} (\Psi + f C \overline{\Psi}) \qquad \text{(III-19a)}$$

$$P_2 \Psi = \frac{1}{2} (\Psi - f C \overline{\Psi}) \qquad \text{(III-19b)}$$

where f is a constant phase factor:

$$f^* f = 1 \qquad \text{(III-19c)}$$

There will be no loss of generality if we take $f = 1$ or make the decomposition:

$$\Psi (x) = U(x) + i V (x) \qquad \text{(III-20a)}$$

$$U (x) = P_1 \Psi (x) = \frac{1}{2} (\Psi (x) + C \overline{\Psi}(x)) \qquad \text{(III-20b)}$$

$$V(x) = P_2 \Psi (x) = \frac{1}{2} (\Psi (x) - C \overline{\Psi}(x)) \qquad \text{(III-20c)}$$

The projection operators P_1 and P_2 defined by (III-20) were introduced by Furry[27] and will be called "Furry projection operators", in what follows. They satisfy our conditions (III-15), (III-16) and (III-17) for the operations of projection[30] although we will have either (III-16) or (III-16a) satisfied, according to the type of improper transformations used, in the following way.

If the improper transformations $_i$ are given by:

$$\Psi \longrightarrow f \beta \Psi \quad \text{(space reflexion)} \qquad \text{(III-21)}$$

$$\Psi \longrightarrow i f \gamma_5 C \overline{\Psi} \quad \text{(time inversion I)} \qquad \text{(III-22a)}$$

$$\Psi \longrightarrow i f \gamma_5 C \overline{\Psi}' \quad \text{(time inversion II)} \qquad \text{(III-22b)}$$

where f is equal to ± 1 or $\pm i$, thus we find the following two cases:

α) The phase factor f in (III-21) and (III-22) is given by:

$$f = i \quad \text{or} \quad f = -i \qquad \text{(III-23a)}$$

---

30). We remember here that in this case the homogeneity condition (III-15d) is satisfied only for α real.

In this case the invariance condition (III-16) is satisfied also if $\mathcal{L}$ is an improper transformation $\mathcal{L}_i$:

$$\mathcal{L}_i P_r \Psi = P_r \mathcal{L}_i \Psi \qquad \text{(III-16)}$$

β) If the phase factors of the improper transformations are:

$$f = 1 \quad \text{or} \quad f = -1 \qquad \text{(III-23b)}$$

then we have, instead of (III-16) the property:

$$\mathcal{L}_i P_1 \Psi = P_2 \mathcal{L}_i \Psi ; \mathcal{L}_i P_2 \Psi = - P_1 \mathcal{L}_i \Psi \qquad \text{(III-16b)}$$

In this case the two projection operators P_1 and P_2 exchange their roles under the transformations of the improper Lorentz group.

2) Impossibility of projections formed by taking linear combinations of Ψ and $\gamma_5 C \overline{\Psi}$.

At first sight one could think that the operators Q_1 and Q_2, defined by:

$$\Psi(x) = Y(x) + Z(x) \qquad \text{(III-24a)}$$

$$Y(x) = Q_1 \Psi(x) = \frac{1}{2} (\Psi (x) + \gamma_5 C \overline{\Psi}) \qquad \text{(III-24b)}$$

$$Z(x) = Q_2 \Psi (x) = \frac{1}{2} (\Psi (x) - \gamma_5 C \overline{\Psi}) \qquad \text{(III-24c)}$$

were projection operators. However this is not the case as we find that the basic condition (III-16b), for projection operators, is not satisfied by Q_1 and Q_2 :

$$Q_r Q_s \neq \delta_{rs} Q_r \quad (r,s = 1,2) \qquad \text{(III-25)}$$

This is still true if a linear combination of Ψ and $\gamma_5 \; C \overline{\Psi}$, more general than that in (III-24), is used.

On the other hand we see that the invariance condition (III-16) (or (III-16a)) is satisfied by the decomposition (III-24) of $\psi(x)$. For the restricted Lorentz group this is a consequence of the fact that $\gamma_5 C \overline{\psi}$ transforms then in the same way as ψ. For the improper transformations (III-21), (III-22) we find, in opposition to the case of Furry's projection, that (III-16a) is satisfied if the phase factor f is $\pm i$; or:

$$\mathcal{b}_i Q_1 \psi = Q_2 \mathcal{b}_i \psi ; \mathcal{b}_i Q_2 \psi = - Q_1 \mathcal{b}_i \psi . \qquad \text{(III-16c)}$$

If $f = \pm 1$, however, (III-16) is satisfied.

Also we find that, if m = 0, the second condition of invariance (III-17a) is also satisfied by Q_1 and Q_2 :

$$\gamma^{\mu} \frac{\partial}{\partial x^{\mu}} Q_r \psi(x) = 0. \qquad \text{(III-17b)}$$

Thus we see that, in the case of zero mass, we have, besides the two types of splitting of $\psi(x)$ by the Schrodinger and Furry projection operators, respectively, that given by (III-24) (which is not an operation of projection, however).

C) Expression of a Dirac field $\psi(x)$ in terms of two half fields.

As it was said in sec A. different representations of a Field Theory will be obtained according to the way in which the fields $\psi(x)$ are expressed in terms of two half fields (positive energy fields). These representations will be analysed in the following sections. In order to find the possible ways of expressing $\psi(x)$ in terms of two half fields we start by decomposing $\psi(x)$ with the aid of one of the three types of operations considered in sec. B (the decomposition in Y, Z fields being possible only for the case of zero mass) and then we introduce the appropriate half fields.

For simplicity of notation it is convenient to introduce, besides the antiparticle (charge conjugate) field $\psi'(x)$, given by:

$$\psi'(x) = C\overline{\psi}(x) \;;\; \psi(x) = C\overline{\psi'}(x) \qquad \text{(III-26)}$$

another field, $\psi''(x)$, defined by:

$$\psi''(x) = \gamma_5 C\overline{\psi}(x) \;;\; \psi(x) = -\gamma_5 C\overline{\psi''}(x) \qquad \text{(III-27)}$$

We shall call $\psi''(x)$ the "contraparticle field". It satisfies the Dirac equation with the opposite sign of the mass term of that in the Dirac equation satisfied by $\psi(x)$:

$$(\gamma^\mu \frac{\partial}{\partial x^\mu} + m)\psi(x) = 0 \qquad \text{(III-28a)}$$

$$(\gamma^\mu \frac{\partial}{\partial x^\mu} - m)\psi''(x) = 0 \qquad \text{(III-28b)}$$

It should be observed that the "contraparticle field" is essentially the same as the "antiparticle field" to which it is related by:

$$\psi''(x) = \gamma_5 \psi'(x)$$

It is of immediate verification that the operators for the number of contraparticles:

$$N'' = i\int d\sigma_\mu \overline{\psi''_+}(x)\gamma^\mu \psi''_+(x)$$

is equal to that for the number of antiparticles:

$$N' = i\int d\sigma_\mu \overline{\psi'_+}(x)\gamma^\mu \psi'_+(x) ,$$

or:

$$N'' = N'.$$

We find the six following possible cases:

1) Expansion into particle and antiparticle half fields.

This is the usual case, given in the example in sec. A.

Here, after performing a decomposition by the Schrodinger projection operators:

$$\psi = \psi_+ + \psi_- \qquad \text{(III-29a)}$$

we express ψ_- in terms of the positive energy part of the antiparticle field:

$$\psi_- = C\overline{\psi'_+} \qquad \text{(III-29b)}$$

Thus all field quantities will be expressed in terms of $\psi_+(x)$, $\psi'_+(x)$, $\psi_+(x)^\dagger$ and $\psi'_+(x)^\dagger$.

2) Decomposition into particle and contraparticle half fields.

After the same decomposition (III-29a) as in case 1) we express ψ_- as:

$$\psi_- = -\gamma_5\, C\overline{\psi''_+} \qquad \text{(III-30)}$$

where the half field ψ''_+ is the positive energy part of the contraparticle field ψ'', defined by (III-27).

We could also express this type of decomposition saying that the two half fields here used are the positive energy parts of $\psi(x)$ and $\psi'(x) = C\,\overline{\psi}(x)$, respectively.

3) Decomposition into U and V half fields.

This was the one used by Majorana[28] in expressing the positron theory in terms of two Majorana fields.

Here we first decompose $\psi(x)$ by the Furry's projection operators:

$$\psi = U + i\,V \qquad \text{(III-31a)}$$

$$U = \frac{1}{2}(\psi + C\overline{\psi})\ ; \quad V = \frac{1}{2}(\psi - C\overline{\psi}) \qquad \text{(III-31b)}$$

and then, using the properties:

$$U = C\,\overline{U}\ ; \quad V = C\,\overline{V} \qquad \text{(III-31c)}$$

we express U and V as:

$$U = U_+ + C\,\overline{U_+}\ ; \quad V = V_+ + C\,\overline{V_+} \qquad \text{(III-31d)}$$

$$V = V_+ + C\,\overline{V_+} \qquad \text{(III-31e)}$$

In (III-31d) the half fields U_+, V_+ are the positive energy parts of U and V, respectively.

4) Decomposition into U and V'' half fields.

Here, after the expansion (III-31a) is made we express V in terms of the corresponding contraparticle field:

$$V'' = \gamma_5 C\,\overline{V}\ , \qquad \text{(III-32a)}$$

or, in view of (III-31c):

$$V'' = \gamma_5\,V'\ ; \quad V' = \gamma_5\,V'' \qquad \text{(III-32b)}$$

Thus we express U and V as :

$$U = U_+ + C\,\overline{U_+} \qquad \text{(III-32c)}$$

$$V = \gamma_5\,V''_+ - \gamma_5 C\,\overline{V''_+} \qquad \text{(III-32d)}$$

5) Decomposition into Y and Z half fields.

In the case when the rest mass is zero we may use the decomposition of ψ (x) given by (III-24):

$$\psi = Y + Z \qquad \text{(III-34a)}$$

$$Y = \frac{1}{2}(\psi + \gamma_5 C \overline{\psi}) \qquad \text{(III-34b)}$$

$$Z = \frac{1}{2}(\psi - \gamma_5 C \overline{\psi}) \qquad \text{(III-34c)}$$

The fields Y and Z are related by:

$$Z = - \gamma_5 C \overline{Y} ; \qquad Y = + \gamma_5 C \overline{Z} \qquad \text{(III-34d)}$$

Now, we have to express all the field operators in terms of the half fields Y_+ and Z_+ (and their hermitian conjugate), which are, respectively, the positive energy parts of Y and Z. We find that:

$$Y_- = + \gamma_5 C \overline{Z}_+ ; \quad Z_- = - \gamma_5 C \overline{Y}_+ \qquad \text{(III-34e)}$$

or:

$$Y = Y_+ + \gamma_5 C \overline{Z}_+ ; \quad Z = Z_+ + \gamma_5 C \overline{Y}_+ \qquad \text{(III-34f)}$$

6) Decomposition into Y and Y' half fields.

Y' is, in virtue of (III-34d) related to Y by:

$$Y' = C \overline{Y} = - \gamma_5 Z ; \quad Z = - \gamma_5 Y' \qquad \text{(III-35a)}$$

Here we start with the decomposition (III-34a) as in the case 6) and then substitute Z by $- \gamma_5 Y'$:

$$\psi = Y - \gamma_5 Y' \qquad \text{(III-35b)}$$

Now we express Y and Y' in terms of the corresponding half fields:

$$Y = Y_+ + C \overline{Y}'_+ ; \quad Y' = Y'_+ + C \overline{Y}_+ \qquad \text{(III-35c)}$$

The substitution of (III-35c) into (III-35b) gives the expression of $\Psi(x)$ in terms of the half fields Y_+ and $Y_+^\dagger$.

D) Configuration representation.

In the field theories, if we find a representation of the field operators then the Schrödinger equation (see part I) can be written, instead of in operational form, as a system of equations (infinite in number) relating the components of the wave function; this is what is called the configuration representation. If in particular the operators for the number of the several kind of particles is diagonal in the representation considered the underlying Hilbert space is called Fock space[31]. On the other hand we may work in the space of the quanta, using operational calculus and taking advantage of the commutation relations of the operators and of the completeness of the basic set of wave functions. This simplifies the computations, in general; for this reason we shall use, in general, this last method, although it will be convenient for the analysis of the structure of the underlying Hilbert space to examine the configuration representation of the theory.

Here we can either look for a representation of the field operators including their dependence on the coordinates (continuous variables) and spinor indices (discrete), in which case we are lead to the coordinate configuration representation of the theory (the wave functions depending on the coordinates), or we make a Fourier expansion of the operators and find a representation of the operators depending on the momentum (continuous) and spin coordinates (discrete); this leads to the momentum configuration representation. For the following analysis it will be simpler to consider the momentum representation, although the method here used can be extended in order to obtain the coordinate representation.

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31). V. Fock, Zeits. f. Phys. 75, 622, 1932.

For the sake of simplicity, as we are in this work mainly concerned with Fermi type of interactions, we restrict the present considerations to the case of four spinor fields $\psi^{(r)}(x)$ $(r = 1,2,3,4)$, corresponding to the four particles involved in the interaction. The extension to the most general case of any number of Fermi and Boson fields interacting with each other is immediate.

For concreteness we expand these fields in plane waves in the usual way (particle - antiparticle representation):

$$\psi^{(r)} = \psi^{(r)}_1 + C\,\overline{\psi^{(r)}_2} \;, \quad \psi'^{(r)} = C\,\overline{\psi^{(r)}} = \psi^{(r)}_2 + C\,\overline{\psi^{(r)}_1} \qquad \text{(III-36)}$$

where

$$\psi^{(r)}_{\rho,\alpha}(x) = \frac{1}{(2\pi)^{3/2}} \sum_{s=1}^{2} \int d_3k\, u^{(r)}_{\alpha}(\vec{k},s)\, e^{ik_\mu x^\mu}\, \psi^{(r)}_{\rho}(\vec{k},s) \quad (\rho = 1,2) \qquad \text{(III-37)}$$

where:

$$k_\mu k^\mu - M^2_{(r)} = 0 \quad . \qquad \text{(III-37a)}$$

s is the dichotomic spin variable (the values 1 and 2 corresponding to the two possible orientations of the spin) and ρ is the half field indice, its values 1 and 2 corresponding respectively to the particle and antiparticle fields in the present case. The spinor functions $u^{(r)}_{\alpha}(k,s)$, for $s = 1,2$ are the two <u>positive energy</u> solutions of the Dirac equation:

$$(i\,\gamma^\mu k_\mu + M_{(r)})\; u^{(r)}(k,s) = 0 \qquad \text{(III-38)}$$

The generalization of (III-36,37) for the case of any type of splitting of the field $\psi(x)$ is obvious.

For simplicity of notation we shall denote the variables $(\vec{k},s)$ by ξ and the indices r and ρ by η:

$$\psi^{(r)}_{\rho}(\vec{k},s) \longrightarrow \psi^{\eta}(\xi) \qquad \text{(III-39a)}$$

also we shall use the notation:

$$\sum_{s=1}^{2} \int d_3 k \longrightarrow \int d\xi \qquad \text{(III-39b)}$$

Thus the anticommutation relations for the half fields can be written:

$$\{\psi^{\eta}(\xi), \psi^{\eta'\dagger}(\xi')\} = \delta(\xi-\xi')\,\delta_{\eta\eta'} \qquad \text{(III-40a)}$$

$$\{\psi^{\eta}(\xi), \psi^{\eta'}(\xi')\} = \{\psi^{\eta\dagger}(\xi), \psi^{\eta'\dagger}(\xi')\} = 0 \qquad \text{(III-40b)}$$

Now a representation of the operators $\psi^{\eta}(\xi)$ could be characterized by the fact that a given complete set of observables should be diagonal in this representation. For instance, if we impose that all the commuting quantities:

$$N^{\eta}(\xi) = \psi^{\eta\dagger}(\xi)\,\psi^{\eta}(\xi), \qquad \text{(III-41)}$$

operators for the number of particles of the type η in the state characterized by ξ, should be diagonal, a special representation could be obtained by a trivial generalization of the one considered by Jordan and Wigner. However, in order to pass to the configuration representation in Fock space, it is not even necessary to use a special representation of the operators $\psi^{\eta}(\xi)$, as long as one is sure that the operators for the number of particles $N^{\eta}(\xi)$ are diagonal.

This is the case if the following method is used:

We first express the quantum wave function $\Psi(t)$ of the Schrödinger representation as[32]:

$$\Psi(t) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \sum_{\eta_1 \cdots \eta_n} \int \cdots \int d\xi_1 \cdots d\xi_n\, \psi_{\eta_1}(\xi_1)^{\dagger} \cdots$$

$$\cdots\, \psi_{\eta_n}(\xi_n)^{\dagger}\; \varphi_{\eta_1 \cdots \eta_n}(\xi_1 \cdots \xi_n, t)\,\Omega_0 \quad . \qquad \text{(III-42)}$$

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32). R. Becker and G. Liebfried, Phys. Rev. 69, 341, 1946.

In (III-42) Ω_o is the vacuum wave function characterized by the condition:

$$\psi_\eta(\xi)\,\Omega_o = 0 \qquad \text{(III-43)}$$

The time dependent functions φ, anti-symmetric in the permutation of any two pairs of variables (η_i, ξ_i), (η_j, ξ_j) are the wave functions in configuration (momentum) space.

The normalization condition for $\Psi(t)$:

$$(\Psi(t), \Psi(t)) = 1 \qquad \text{(III-44a)}$$

can be expressed, by the use of (III-42), in configuration space as:

$$\varphi_o^\dagger\varphi_o + \sum_{n=1}^{\infty} \sum_{\eta_1 \ldots \eta_n} \int \cdots \int \varphi_{\eta_1 \ldots \eta_n}(\xi_1 \ldots \xi_n; t)^\dagger \varphi_{\eta_1 \ldots \eta_n}(\xi_1 \ldots \xi_n; t)\, d\xi_1 \ldots d\xi_n$$
$$\text{(III-44a)}$$

In obtaining (III-44a), we used the anticommutation relations (III-40) and the property (III-43) of the vacuum wave function Ω_o.

In order to show that the operators $N^\eta(\xi)$ given by (III-41) are now automatically diagonal we observe that the quantum wave functions $\Psi(t)$ given by (III-42) are expanded by the complete set of orthonormal vectors:

$$\Omega_{\eta_1 \ldots \eta_n}(\xi_1 \ldots \xi_n) = \frac{1}{(n!)^{3/2}} \sum (-1)^p P \left[\psi_{\eta_1}(\xi_1)^\dagger \ldots \psi_{\eta_n}(\xi_n)^\dagger\right]\Omega_o \qquad \text{(III-45)}$$

where:

$$n = 0,1,2,3, \ldots \qquad \text{(III-45a)}$$

and P is the operator of permutation of the variables (η_i, ξ_i), p being the order of the permutation.

These vectors (III-45) are eigenfunctions of the operators $N^\eta(\xi)$ with eigenvalues 1 or 0 (when the continuous variables $\vec{k}$ are substituted by discrete ones in the usual way). They are also eigenfunctions of the operators for the numbers of particles (or antiparticles) of the several types:

$$N^{\eta} = \int d\xi \; N^{\eta}(\xi), \qquad \text{(III-46)}$$

with eigenvalues 0,1,2,3, ... ;

and of that for the total number of particles:

$$N = \sum_{\eta} N^{\eta}, \qquad \text{(III-47)}$$

with the eigenvalue n. The antisymmetrization of the product of operators $\psi^{\eta}(\xi)^{\dagger}$ in the expression (III-45) is an immediate consequence of the anti-commutation relation (III-40b), in virtue of which the symmetrical part of

$$\psi_{\eta_1}(\xi_1)^{\dagger} \; \psi_{\eta_2}(\xi_2)^{\dagger}$$

vanishes. In the expression (III-42) for the wave function $\Psi(t)$ the anti-symmetrization of the product of operators is taken care of by the condition that φ be antisymmetric; one sees thus that the usual antisymmetrization of the wave function in configuration space is intimately related to the anti-commutation properties of the field operators.

Now the Schrödinger equations in configuration representation can be obtained by substituting in the quantum Schrodinger equation (see part I) $\Psi(t)$ by the expression (III-42) and taking the scalar product with all the vectors of the type (III-45), for $n = 0,1,2,3, \ldots$.

E) Structure of the underlying Hilbert space of wave functions:

In the above treatment we made only a slight distinction between the dependence of the wave functions on the variables $\xi_i = \vec{R}_i, s_i$ which were treated as coordinates, and the indices $\eta_i = r_i, \rho_i$ which indicated the different types of particles we are concerned with. The convenience of treating the index ρ in the same footing as r comes from the fact that particles and antiparticles really behave as different kinds of particles. Although the index η could have been included in ξ as a coordinate (actually in the anti-symmetrization of φ the index η and the coordinates ξ are treated similarly) it is convenient to make a clear cut distinction of the roles of ξ and η and then factorize the Hilbert space into the direct product of several subspaces corresponding to the different types of particles characterized by the several values of η.

Also, as seen before, there are ways of splitting the fields $\psi^{(r)}$ other than the usual one given in the example of sec. D, say into particle and antiparticle fields. These several possibilites lead to different representations of the theory, which are however, related by unitary transformations and are thus equivalent; nevertheless it is convenient, for the sake of simplicity of the computations, to use in some cases one or another of these representations. The present analysis of the structure of the Hilbert space is just the consideration of these possible representations and the factorization of the total Hilbert space into the direct product of independent sub-spaces.

As it will be clear in the analysis of the theories of neutrinos (Part IV) an understanding of the underlying Hilbert space of wave functions is essential for the comprehension of the physical distinction between different kinds of neutral particles.

We consider first a general case of factorization of the Hilbert space for Dirac fields.

a) Factorization of the Hilbert space (general treatment):

If we decompose in a relativistic invariant way the field quantities $\psi^{(r)}(x)$ and $\overline{\psi^{(r)}}(x)$ in terms of two other half fields (positive energy fields) $\psi^{(r,\rho)}(x) = \psi^{\eta}(x)$ $(\rho = 1,2)$, satisfying the same Dirac equation (in interaction representation) as $\psi^{(r)}(x)$, and their hermitian conjugates, using any one of the procedures referred to before, it is convenient to use a representation in which all the operators $\psi_{\eta}(\xi)$ are factorized as a direct product of several independent (operating in different spaces) half fields $\phi^{\eta}(\xi)$; to this correspond a factorization of the Hilbert space.

α) We shall consider first, for the sake of simplicity, the case when we have only two such anticommuting half fields $\psi^{\eta}(\xi)$ ($\eta = 1,2$). We introduce the notation:

$$\psi^{\eta}(\xi_1 \ldots \xi_{N_\eta}) = \frac{1}{(N_\eta !)^{3/2}} \sum_P (-1)^P P \left[\psi^{\eta}(\xi_1)^{+} \ldots \psi^{\eta}(\xi_{N_\eta})^{+} \right] ,$$

$$\eta = 1,2 \qquad \text{(III-48a)}$$

Now we see that the basic vector $\Omega_{\eta_1 \ldots \eta_m}(\xi_1 \ldots \xi_m)$ given by (III-45) can be expressed in terms of the quantities (III-48a) as:

$$\Omega_{\eta_1 \ldots \eta_N}(\xi_1 \ldots \xi_N) = \pm \Omega(\xi^1_1 \ldots \xi^1_{N_1} ; \xi^2_1 \ldots \xi^2_{N_2}) \qquad \text{(III-49)}$$

where:

$$\Omega(\xi^1_1 \ldots \xi^1_{N_1} ; \xi^2_1 \ldots \xi^2_{N_2}) = \psi^1(\xi^1_1 \ldots \xi^1_{N_1}) \psi^2(\xi^2_1 \ldots \xi^2_{N_2}) \Omega_0 ,$$

$$N_1 + N_2 = N \qquad \text{(III-49a)}$$

in the case when η assumes only two values: 1 or 2 (i.e. for only one Fermi field).

In the first member of (III-49) the particle coordinates ξ_i are enumerated from 1 to N, regardless of the type of particle (which is indicated by the value of the corresponding index η_i). In the second member of (III-49) they are enumerated from 1 to N_1 for the first type of particle and from 1 to N_2 for the second type. The $\pm$ sign in (III-49), which depends on the ordering of the particles in the first member, is unessential and shall be incorporated on the configuration wave function φ in the following step.

Using (III-49) and (III-49a) we now express the total wave function (III-45) as:

$$\Psi = \sum_{N_1, N_2 = 0}^{\infty} \int \cdots \int d\xi_1^1 \cdots d\xi_{N_1}^1 d\xi_1^2 \cdots d\xi_{N_2}^2 \, \Omega(\xi_1^1 \cdots \xi_{N_1}^1, \xi_1^2 \cdots \xi_{N_2}^2)$$

$$\varphi(\xi_1^1 \cdots \xi_{N_1}^1, \xi_1^2 \cdots \xi_{N_2}^2) \qquad \text{(III-50)}$$

Here the wave function φ has to be anti-symmetric only in the coordinates of the same type of particles. The upper index in ξ_i^η corresponds to the type of particle and the lower one to the numbering of them among the particles of the same type.

Let us call $\mathcal{A}$ the algebra generated by the operators $\psi^1(\xi)$, $\psi^2(\xi)$ (and their hermitian conjugates) by products and linear combinations. The operators of $\mathcal{A}$ act on the Hilbert space $\mathcal{G}$.

The operators ψ^1 and ψ^2 satisfy the following anticommutation relations:

$$\{\psi^\eta(\xi), \psi^{\eta'}(\xi')^\dagger\} = \delta_{\eta\eta'} \delta(\xi - \xi') \qquad \text{(III-51)}$$

All the other anticommutators vanish.

Now we consider also the algebra $\mathcal{A}^1$ generated by the operators $\phi^1(\xi)$, which have the same commutation relations as $\psi^\eta(\xi)$

$$\{\phi^1(\xi), \phi^{1\dagger}(\xi')\} = \delta(\xi - \xi') \qquad \text{(III-52a)}$$

$$\{\phi^1(\xi), \phi^1(\xi')\} = \{\phi^{1\dagger}(\xi), \phi^{1\dagger}(\xi')\} = 0 \qquad \text{(III-52b)}$$

but operates on a vector space $\mathcal{G}_1$ with the square root of the dimension of that of $\mathcal{G}$. The vector space $\mathcal{G}^1$ is expanded by the complete orthonormal set of vectors:

$$\Omega^1(\xi_1^1 \cdots \xi_{N_1}^1) = \phi^1(\xi_1^1 \cdots \xi_{N_1}^1)\Omega_0^1, \; N_1 = 0, 1, 2, 3, \ldots \qquad \text{(III-53)}$$

Ω_o^1 being defined by:

$$\phi^1(\xi)\Omega_o^1 = 0 \qquad \text{for all } \xi \text{ 's}, \qquad \text{(III-54)}$$

and

$\phi^1(\xi_1^1 \ldots \xi_{N_1}^1)$ by:

$$\phi^\eta(\xi_1 \ldots \xi_{N_\eta}) = \frac{1}{(N_\eta)^{3/2}} \sum_P (-1)^{P_P} \left[\phi^\eta(\xi_1)^\dagger \ldots \phi^\eta(\xi_{N_\eta})^\dagger \right. .$$

$$(\eta = 1,2) \qquad \text{(III-48b)}$$

Also we consider another algebra $\mathcal{A}^2$, isomorphic to the first one, generated by operators $\phi^2(\xi)$, with the same commutation relations as those for $\phi^1(\xi)$, but acting on a different vector space $\mathcal{G}^2$. $\mathcal{G}^2$ is expanded by the base vectors:

$$\Omega^2(\xi_1^2 \ldots \xi_{N_2}^2) = \phi^2(\xi_1^2 \ldots \xi_{N_2}^2)\Omega_o^2 , \quad N_2 = 0,1,2,\ldots \quad \text{(III-55)}$$

Ω_o^2 being defined by:

$$\phi_2^2(\xi)\Omega_o^2 = 0 \quad \text{for all } \xi \text{ 's}. \qquad \text{(III-56)}$$

$\phi^2(\xi_1^2 \ldots \xi_{N_2}^2)$ is defined by (III-48b) with $\eta = 2$.

We want to factorize $\mathcal{G}$ as a direct product of $\mathcal{G}_1$ and $\mathcal{G}_2$:

$$\mathcal{G} = \mathcal{G}^1 \times \mathcal{G}^2 \qquad \text{(III-57)}$$

In other words, we want to express:

$$\Omega_o = \Omega_o^1 \times \Omega_o^2 \qquad \text{(III-58)}$$

$$\Omega_2(\xi_1^1 \ldots \xi_{N_1}^1 ; \xi_1^2 \ldots \xi_{N_2}^2) = \psi^1(\xi_1^1 \ldots \xi_{N_1}^1) \cdot \psi^2(\xi_1^2 \ldots \xi_{N_2}^2)(\Omega_o^1 \times \Omega_o^2) =$$

$$= \Omega^1(\xi_1^1 \ldots \xi_{N_1}^1) \times \Omega^2(\xi_1^2 \ldots \xi_{N_2}^2) . \qquad \text{(III-59)}$$

Also we want to factorize the algebra $\mathcal{A}$ as a direct product of $\mathcal{A}^1$ by $\mathcal{A}^2$,

$$\mathcal{A} = \mathcal{A}^1 \times \mathcal{A}^2 \qquad \text{(III-60)}$$

Expression (III-59) will become an identity if we take for the connexion of the operators $\psi^1(\xi), \psi^2(\xi)$ on $\mathcal{G}$ and $\psi^q(\xi)$ on $\mathcal{G}^q$ the usual relations:

$$\psi^1(\xi) = \phi^1(\xi) \times 1 \;;\; \psi^2(\xi) = 1 \times \phi^2(\xi) \qquad \text{(III-61)}$$

where the symbol x indicates the operation of direct product. Here, however, we cannot assume these relations because the operators $\psi^1(\xi)$ and $\psi^2(\xi)$ would in such a case <u>commute</u> with each other, instead of anticommute, as it is the actual case. In order to get the appropriate anticommutation relation between these half fields, we may take, for instance, instead of (III-61)

$$\psi^1(\xi) = \phi^1(\xi) \times 1 \;;\; \psi^2(\xi) = R^1 \times \phi^2(\xi) \qquad \text{(III-62)}$$

where R^1 is a unitary element of $\mathcal{A}_1$ with the property:

$$R^1 \Omega^1_0 = \Omega^1_0 \,, \qquad \text{(III-63)}$$

which anticommutes with all $\phi^1(\xi)$ and $\phi^1(\xi)^\dagger$:

$$\{R^1, \phi^1(\xi)\} = \{R^1, \phi^1(\xi)^\dagger\} = 0 \qquad \text{(III-64)}$$

In the Jordan-Wigner representation, for instance, R^1 is the element of $\mathcal{A}^1$ of the type:

$$R^1 = \beta^1(\xi_1) \times \beta^1(\xi_2) \times \cdots \beta^1(\xi_i) \times \ldots = R^{1\dagger} \qquad \text{(III-65)}$$

where $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (III-65a)

Another possible representation corresponds to take for R^1 the operator:

$$R^1 = (-1)^{N(1)} = e^{i\pi N(1)} = 1 + i\pi N_{(1)} - \frac{\pi^2}{2!} N^2_{(1)} \cdots , \qquad \text{(III-66)}$$

$$N_{(1)} = \int d\xi \, \phi^1(\xi)^\dagger \phi^1(\xi) = \int d\xi \, N_{(1)}(\xi) \qquad \text{(III-66a)}$$

as it satisfies our conditions of unitarity and anticommutation:

$$R^{+} = R^{-1} \; ; \quad \left\{ (-1)^{N_{(1)}}, \phi^{+}(\xi) \right\} = \left\{ (-1)^{N_{(1)}}, \phi^{1}(\xi)^{+} \right\} = 0 \quad \text{(III-67)}$$

as well as condition (III-63).

Now we find that the operators $N^1(\xi)$ and $N^2(\xi)$ defined by:

$$N^{\eta}(\xi) = \psi^{\eta}(\xi)^{+} \psi^{\eta}(\xi), \; (\eta = 1,2) \qquad \text{(III-68)}$$

are related to $N_{(\;)}$ by:

$$N^1(\xi) = N_{(1)}(\xi) \times 1; \quad N^2(\xi) = 1 \times N_{(2)}(\xi) \qquad \text{(III-68a)}$$

Thus we can write:

$$\psi^1(\xi) = \phi^1(\xi) \times 1 \; ; \psi^2(\xi) = (-1)^{N(1)} \times \phi^2(\xi) =$$

$$= (-1)^{N(1)} (1 \times \phi^2(\xi)) \qquad \text{(III-69}$$

It is now easy to verify that:

$$\Omega(\xi^1_1 \ldots \xi^1_{N_1} ; \xi^2_1 \ldots \xi^2_{N_2}) = \pm \phi^1(\xi^1_1 \ldots \xi^1_{N_1})(-1)^{N(1)} \times \phi^2(\xi^2_1 \ldots \xi^2_{N_2})(\Omega^1_0 \times \Omega^2_0)$$

or, using the property:

$$(-1)^{N(1)} \Omega^1_0 = \Omega^1_0 \qquad \text{(III-70)}$$

(compare (III-63)) we finally find the factorization:

$$\Omega(\xi^1_1 \ldots ; \xi^2_1 \ldots) = \pm \Omega^1(\xi^1_1 \ldots \xi^1_{N_1}) \times \Omega^2(\xi^2_1 \ldots \xi^2_{N_2}) \quad \text{(III-71)}$$

as needed.

Thus, summarising, we can factor the algebra $\mathcal{A}$ and the underlying Hilbert space $\mathcal{G}$ as:

$$\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \; ; \; \mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \qquad \text{(III-72)}$$

if the connection between the generators $\psi^1(\xi)$ and $\psi^2(\xi)$ of $\mathcal{A}$ and those $\phi^1(\xi)$ and $\phi^2(\xi)$ of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, are taken as:

$$\psi^{1}(\xi) = \phi^{1}(\xi) \times 1; \quad \psi^{2}(\xi) = R^{1} \times \phi^{2}(\xi) \qquad \text{(III-73)}$$

the operator R^1 having the properties referred to above.

β) In general, if we have q Dirac fields $\psi_{(r)}(x)$ a similar factorization can be done:

Here there will be 2q anticommuting half fields:

$$\psi^{1}_{(1)}(\xi),\ \psi^{2}_{(1)}(\xi),\ \ldots\ldots\ldots,\ \psi^{1}_{(q)}(\xi),\ \psi^{2}_{(q)}(\xi)$$

operating on a vector space $\mathcal{G}$ and generating the algebra $\mathcal{A}$; it is not necessary that the decomposition of two different fields $\psi_{(r)}(x)$ and $\psi_{s}(x)$ into the corresponding half fields $\psi^{1}_{(r)}(\xi)$, $\psi^{2}_{(r)}(\xi)$ and $\psi^{1}_{s}(\xi)$, $\psi^{2}_{s}(\xi)$ be of the same type. For instance we could use for some of them the usual decomposition into particle and antiparticle half fields and for others into two Majorana fields in the way that will be analysed in section b.

Now if we consider the 2q independent fields:

$$\phi^{1}_{(1)}(\xi),\ \phi^{2}_{(1)}(\xi),\ \ldots\ldots\ldots,\ \phi^{1}_{(q)}(\xi),\ \phi^{2}_{(q)}(\xi)$$

acting respectively in the subspaces:

$$\mathcal{G}^{1}_{(1)},\ \mathcal{G}^{2}_{(1)},\ \ldots\ldots\ldots\ldots,\ \mathcal{G}^{1}_{(q)},\ \mathcal{G}^{2}_{(q)}$$

and generating respectively the algebras:

$$\mathcal{A}^{1}_{(1)},\ \mathcal{A}^{2}_{(1)},\ \ldots\ldots\ldots,\ \mathcal{A}^{1}_{(q)},\ \mathcal{A}^{2}_{(q)}$$

which are, as before, isomorphic to each other, we will have again the factorization:

$$\mathcal{G} = \mathcal{G}^{1}_{(1)} \times \mathcal{G}^{2}_{(1)} \times \cdots\cdots \times \mathcal{G}^{1}_{(q)} \times \mathcal{G}^{2}_{(q)} \qquad \text{(III-74)}$$

$$\mathcal{A} = \mathcal{A}^{1}_{(1)} \times \mathcal{A}^{2}_{(1)} \times \cdots\cdots \times \mathcal{A}^{1}_{(q)} \times \mathcal{A}^{2}_{(q)} \qquad \text{(III-75)}$$

if we take:

$$\psi^{1}_{(r)}(\xi) = R^{1}_{(1)} \times R^{2}_{(1)} \times \ldots\ldots\times R^{2}_{(r-1)} \times \phi^{1}_{(r)}(\xi) \times 1^{2}_{(r)} \times \ldots \times 1^{2}_{(q)} \tag{III-76}$$

$$\psi^{2}_{(r)}(\xi) = R^{1}_{(1)} \times R^{2}_{(1)} \times \ldots\ldots\times R^{1}_{(r)} \times \phi^{2}_{(r)}(\xi) \times 1^{1}_{(r+1)} \times \ldots \ldots \times 1^{2}_{(q)} \tag{III-77}$$

where $r = 1,2, \ldots q$, and the unitary operators $R_{(r)}$ satisfy the obvious generalizations of the conditions (II-63) and (III-64).

In the generalization of the Jordan-Wigner representation all of these operators $R_{(r)}$ have the form (III-65).

In the second type of representation referred to above we should take:

$$R^{(\rho)}_{(r)} = (-1)^{N^{(\rho)}_{(r)}}, \quad r = 1,2, \ldots q \,;\ \rho = 1,2. \tag{III-78}$$

where: $$N^{(\rho)}_{(r)} = \int d\xi\, \phi^{\rho}_{(r)}(\xi)^{+} \phi^{\rho}_{(r)}(\xi) \tag{III-78a}$$

Here, instead of (III-68a), the operators for the number of particles corresponding to the several half fields $\psi^{\rho}_{(r)}(\xi)$ will be given by:

$$N^{1}_{(r)} = 1^{1}_{(1)} \times \ldots\ldots \times N^{(1)}_{(r)} \times 1^{2}_{(r)} \times \ldots \times 1^{2}_{(q)} \tag{III-79}$$

$$N^{2}_{(r)} = 1^{1}_{(1)} \times \ldots\ldots \times 1^{1}_{(r)} \times N^{2}_{(r)} \times \ldots\ldots \times 1^{2}_{(q)} \tag{III-80}$$

Finally, corresponding to the relations (III-69) we have now:

$$\psi^{\eta}(\xi) = (-1)^{\sum_{\eta'<\eta} N^{\eta'}} \left[1^{1} \times 1^{2} \times \ldots \times \phi(\xi) \times 1^{\eta+1} \times \ldots \times 1^{2q} \right] \tag{III-81}$$

where $\eta = 1,2,3,\ldots, 2q$ correspond to the pair of indices (r,ρ) enumerated in the order as they appear in (III-76) and (III-77).

b) Different representations resulting from the type of decomposition used for the several fields $\psi_{(r)}(x)$:

As we have said before, there are several different ways of making a relativistic invariant decomposition of the fields $\psi_{(r)}(x)$ and introducing the two half fields associated with each value of r; they will be more carefully analysed in what follows. The use for the decomposition of the several fields (r = 1,2, ...q) of one of these procedures (not necessarily the same for all r's) lead to different representations of the theory. Here we consider, for simplicity, the case of only one field $\psi(x)$, the extension to the most general case being obvious.

According to the analysis in sec. C there are, in the case of arbitrary mass, four different ways of splitting the total field $\psi(x)$ into two half fields $\psi^1(x)$, $\psi^2(x)$, in a relativistic invariant way, and two more possibilities if m = 0. ψ^1 and ψ^2 satisfy the same Dirac equation as ψ:

$$(\gamma^\mu \frac{\partial}{\partial x^\mu} + M_r)\psi^r(x) = 0, \quad (r = 1,2),$$

but eventually with a change in the sign of the mass term (in which case we call the correspondent field a contraparticle field). These decompositions are the following (the first four being possible for any value of the mass, the last two only if m = 0).

1) Expansion in $\psi_+ \psi'$ (particle and antiparticle) fields.

This is the usual splitting in which the total field is decomposed into the positive and the negative energy parts, the last one being expressed in terms of a half field $\psi'_+(x)$ (antiparticle field) by the relation:

$$\psi_-(x) = C\,\overline{\psi'_+(x)} \qquad \text{(III-82)}$$

Now, if we expand $\psi_+(x)$ and $\psi'_+(x)$ in terms of $\psi(\xi)$ and $\psi'(\xi)$ by expressions (III-37) we find (in interaction representation) for the energy-momentum four vector[33]:

$$P_\mu = \int d\xi \;\; k_\mu \left[\psi(\xi)^\dagger \psi(\xi) + \psi'(\xi)^\dagger \psi'(\xi) \right] \qquad \text{(III-83)}$$

where:

$$k_o = + \sqrt{\vec{k}^2 + m^2} . \qquad \text{(III-83a)}$$

The operator for the total number of particles is:

$$N = \int d\xi \; \psi(\xi)^\dagger \psi(\xi) \qquad \text{(III-84a)}$$

and that for the number of antiparticles:

$$N' = \int d\xi \; \psi'(\xi)^\dagger \psi'(\xi) \qquad \text{(III-84b)}$$

If the field is charged the total charge is given by:

$$Q = - e \int d\xi \left[\psi(\xi)^\dagger \psi(\xi) - \psi'(\xi)^\dagger \psi'(\xi) \right] \qquad \text{(III-85)}$$

In the present case as the set of observables:

$$N(\xi) = \psi(\xi)^\dagger \psi(\xi) \qquad \text{(III-86)}$$

$$N'(\xi) = \psi'(\xi)^\dagger \psi'(\xi) , \qquad \text{(III-87)}$$

their eigenvalues being 0 or 1 (when the continuous variables $\xi = (\vec{k}, s)$ are replaced by discrete ones on the usual way), are diagonal both P_μ and Q become automatically diagonal. This permits us to attribute to a state with m particles and n antiparticles the charge:

$$Q = - e\,(m - n)$$

The factorization of $\mathcal{G}$ and $\mathcal{A}$ can now be carried in the way referred to before, by introducing the half fields $\phi_p(\xi)$ and $\phi_a(\xi)$ operating respectively on $\mathcal{G}_p$ and $\mathcal{G}_a$:

$$\mathcal{G} = \mathcal{G}_p \times \mathcal{G}_a \; ; \; \mathcal{A} = \mathcal{A}_p \times \mathcal{A}_a \qquad \text{(III-88)}$$

33). See, for instance, G. Wentzel, Quantum Theory of Fields, 1949. Interscience Publishers, New York.

Symmetry between particle and antiparticle:

Consider two cases:

a) The interaction is bilinear in $\psi(x)$ (and $\psi^{+}(x)$. Consider for instance the case of quantum electrodynamics. Here if we make the transformation $\psi \longrightarrow C\,\overline{\psi}$ and at the same time change the sign of the charge, all the equations are left unchanged; thus the exchange of the roles of particles and antiparticles is equivalent to the change of the sign of the charge - in other words, if we associate the particle field to positrons (instead of negatons) and the antiparticle field to negatons, but change the sign of the charge, the theory will remain isomorphic to itself.

b) The interaction is linear in $\psi(x)$. Consider, for instance, the case of the Fermi interaction of the type:

$$\mathcal{H} = g\,\overline{\psi}_P(x)\,\psi_N(x)\,\overline{\psi}_e(x)\,\psi_\nu(x) + \text{h.c.} \qquad \text{(III-89)}$$

Here we fix our attention on the behaviour of the neutrino field. Now if we change ψ_ν into $C\,\overline{\psi}_\nu$ we obtain a theory where the term (III-89), in the interaction, will be substituted by:

$$\mathcal{H}' = g\,\overline{\psi}_P\,\psi_N\,\overline{\psi}_e\,C\,\overline{\psi}_\nu + \text{h.c.} \qquad \text{(III-90)}$$

If we expand ψ and $C\,\overline{\psi}_\nu$ in plane waves and compare corresponding terms we see that the roles of the neutrino and antineutrino half fields $\psi(\xi)$ and $\psi'(\xi)$ are interchanged in these theories. It is convenient then to factorize the neutrino field in the second case as:

$$\mathcal{A} = \mathcal{A}_a \times \mathcal{A}_\nu \;;\; \mathcal{G} = \mathcal{G}_a \times \mathcal{G}_F \qquad \text{(III-91)}$$

instead of (III-88) (the notation is devious).

However as there is an isomorphism between $\mathcal{G}_a$ and $\mathcal{G}_F$ as well as between $\mathcal{A}_a$ and $\mathcal{A}_{\beta}$, and in view of the fact that $\psi(x)$ and $C\overline{\psi}(x)$ satisfy the same Dirac equation (in interaction representation) the equivalence of the two theories is obvious.

Nevertheless, if we mix terms of type (III-89) and (III-90) in the interaction (say if $\psi(x)$ appears in it in the combination: $\psi(x) + C\overline{\psi}(x)$ we are not anymore allowed to make the substitution:

$$\psi + \lambda\, C\overline{\psi} \longrightarrow (1+\lambda)\,\psi \qquad \text{(III-92)}$$

although if we make the substitution:

$$\psi + \lambda C\overline{\psi} \longrightarrow \lambda\,\psi + C\overline{\psi} \qquad \text{(III-93)}$$

we again obtain a theory isomorphic to the initial one.

For the same reason the following types of Fermi interactions are equivalent:

$$g\,\overline{\psi}_P\,\psi_N\,\overline{\psi}_e\,\gamma_5\,\psi_\nu + \text{h.c.} \qquad \text{(III-94)}$$

$$g\,\overline{\psi}_P\,\psi_N\,\overline{\psi}_e\,\gamma_5\,C\overline{\psi}_\nu + \text{h.c.} \qquad \text{(III-95)}$$

They will be considered in the Part IV.

2) Expansion into ψ, ψ'' fields (particle and contraparticle fields)

As observed before the field quantity $\psi'' = \gamma_5\,C\overline{\psi}$ satisfies the same Dirac equation as ψ, <u>but</u> with a change in the sign of the mass term, we call $\psi''(x)$ a contraparticle field.

If we expand $\psi(x)$ and $\psi''(x)$ (compare (III-30) as:

$$\psi(x) = \psi_+(x) - \gamma_5\, C\,\overline{\psi''_+(x)} \qquad \text{(III-96)}$$

$$\psi''(x) = \psi''_+(x) + \gamma_5\, C\,\overline{\psi_+(x)} \qquad \text{(III-97)}$$

we see that the anticommutators of $\psi''_+(x)$ differ from those for $\psi_+(x)$ only on the sign of the mass term.

If we introduce now the fields $\phi_p(\;)$, $\phi_c(\;)$ (after the expansion in plane waves) we can factorize $\mathcal{G}$ and $\mathcal{A}$ as:

$$\mathcal{G} = \mathcal{G}_p \times \mathcal{G}_c \; ; \; \mathcal{A} = \mathcal{A}_p \times \mathcal{A}_c \qquad \text{(III-97)}$$

Also we should observe that the expressions for the energy-momentum, number of particles (and contraparticles) and total charge are given by expressions (III-83 to 85) where we change:

$$\psi'(\xi) \longrightarrow \psi''(\xi)$$

(Thus the contraparticle field has opposite charge to the partile field).

Surely the representation here obtained is equivalent[34] to the first one, as $\psi'' = \gamma_5 \psi'$; however in certain cases the use of this representation may lead more directly to the needed result. This is the case, for instance, when ψ appears in the interaction in the combination:

$$Y = \psi + \gamma_5 C \overline{\psi} \; ; \qquad \text{(III-99)}$$

for instance, in the interaction hamiltonianof the scalar theory:

$$\mathcal{H}(x) = g \overline{\psi}_P \psi_N \; \psi_e (\psi_\nu + C \overline{\psi}_\nu) + \text{h.c.}$$

These types of theory will be called Fireman theories in Part IV.

Now if we make the expansion (III-96-97):

$$Y = \psi_+ + \psi_+'' + \gamma_5 C (\overline{\psi}_+ - \overline{\psi}_+'') \qquad \text{(III-100)}$$

(notice the assymetry in the signs)

34). The difference between this representation and the first one is more clear in configuration space. In the case (1) we use the same one particle wave function $\psi(x; \vec{k}, s)$ to describe a positon in the state of momentum $\vec{k}$ and spin s as the one used for a negaton in the same state. In the case (2) the referred positon is described by the wave function: $\gamma_5 \psi(x; \vec{k}, s)$ (all these wave functions are of positive energy).

we find that:

a) For first order transitions the lack of symmetry (in the signs) of ψ_+ and ψ_+'' is of no importance because terms in ψ_+ and in ψ_+'' do not combine. However, as the Schrödinger-Casimir projection operators for these fields have different signs in the mass term there will be a cancellation of the contributions of these terms coming from the two types of process: with emission of particle or of contraparticle (as already observed by Fireman[10]). This remains true for the mass depending terms coming from the real emission of neutrinos in processes of any order.

b) For second order processes of the type of double β decay (with no neutrinos) the only contribution is the one coming from the <u>mass term</u> of the Schrödinger projection operator for the neutrino. In other words, there is no double β decay (with no neutrino emitted) in Fireman's mixed theory if the mass of the neutrino vanishes. This is obvious in the present representation because as $\psi_+(x)$ appears in the interaction in the combination $\psi_+(x) + \psi_+''$ and $\overline{\psi}_+(x)$ as $\overline{\psi}_+(x) - \overline{\psi}_+''(x)$ and because the referred process arises from the crossed terms of the type:

$$(\overline{\psi}_P \; \psi_N \overline{\psi} \, Y_\nu) \; (\overline{\psi}_P \, \psi_N \; \overline{\psi} \, Y_\nu)$$

then the contribution to the matrix element arising from transitions where the intermediate neutrino is a contraparticle just cancel the one arising from intermediate particle states. If the mass is not zero then the mass term of the projection operator survives as it has different sign for particle and contraparticle. This contribution is however negligible in relation to that for the decay with two neutrinos in view of the smallness of the neutrino mass. The reason for the different result obtained by Fireman is to be found in the fact that, although for first order transitions he considered the contributions from the two types of particles, for second order he only considered those from the neutrino half field.

3) Expansion into U, V fields (Majorana fields).

Here, for convenience of notation we expand the total field $\Psi(x)$ in terms of U and V defined by:

$$U = \frac{1}{\sqrt{2}} (\Psi + C \overline{\Psi}) = C \overline{U} \qquad \text{(III-101)}$$

$$V = \frac{1}{\sqrt{2}} (\Psi - C \overline{\Psi}) = - C \overline{V} \, , \qquad \text{(III-102)}$$

instead of in terms of the two Furry projections given in sec. C, equations (III-20) (notice the changes by factors $\sqrt{2}$ and i).

Thus, taking the Schrödinger projections, we find for the relation between the U, V half fields and particle, antiparticle half fields[26]:

$$\Psi_+ (x) = \frac{1}{\sqrt{2}} (U_+(x) + V_+(x)) \; ; \Psi^c_+(x) = \frac{1}{\sqrt{2}} (U_+(x) - V_+(x)) \qquad \text{(III-103)}$$

$$U_+ (x) = \frac{1}{\sqrt{2}} (\Psi_+(x) + \Psi^c_+(x)) \; ; \; V_+(x) = \frac{1}{\sqrt{2}} (\Psi_+(x) - \Psi^c_+(x) \qquad \text{(III-104)}$$

Also we find the commutation relations:

$$\{ U_{+\alpha}(x), \overline{U_{+\beta}}(x') \} = \{ V_{+\alpha}(x), \overline{V_{+\beta}(x')} \} = \frac{1}{i} S^{(+)}_{\alpha\beta} (x-x') \qquad \text{(III-105)}$$

all the other anticommutators being zero; in particular:

$$\{ U_+, V_+ \} = \{ U_+ , \overline{V_+} \} = 0 \qquad \text{(III-106)}$$

Thus the expansion of $U_+(x)$ and $V_+(x)$ in terms of $U(\xi)$ and $V(\xi)$ respectively, according to (III-37) leads to the commutation relations:

$$\{ U(\xi), U^\dagger(\xi') \} = \{ V(\xi), V(\xi')^\dagger \} = \delta(\xi - \xi') \qquad \text{(III-107)}$$

all the other anticommutators being zero. These are the same anticommutation relations satisfied by the half fields $\Psi(\xi)$ and $\Psi^c(\xi)$.

Thus the algebra generated by $U(\xi)$, $V(\xi)$ is isomorphic on that generated by $\Psi(\xi)$, $\Psi^c(\xi)$.

Also we find for the number of particles of U and V type, respectively:

$$N_U = \int d\xi \; U(\xi)^\dagger \; U(\xi) \qquad \text{(III-108)}$$

$$N_V = \int d\xi \; V(\xi)^\dagger \; V(\xi) \qquad \text{(III-109)}$$

and for the energy momentum vector:

$$P_\mu = \int d\xi \; k_\mu \left[U(\xi)^+ U(\xi) + V(\xi)^+ V(\xi) \right] \quad \text{(III-110)}$$

(compare with (III-83,84). However, if the field is charged, we find for the total charge:

$$Q = e \int d\xi \left[U(\xi)^+ V(\xi) + V(\xi)^+ U(\xi) \right] \quad \text{(III-111)}$$

Thus we see that we cannot diagonalize simultaneously Q and N_+, N_-. In other words, we cannot attribute a definite charge to a U and V particle. For this reason it is not convenient to use this representation for charged particles. In the applications we will use in general for the charged particles and neutron the first type of representation, reserving the other possibilities for the neutrino.

The factorization of $\mathcal{G}$ and $\mathcal{A}$, as:

$$\mathcal{G} = \mathcal{G}_U \times \mathcal{G}_V \; ; \; \mathcal{A} = \mathcal{A}_U \times \mathcal{A}_V \quad \text{(III-112)}$$

runs in the general lines as before, through the introduction of the independent fields $\phi_U(\xi)$ and $\phi_V(\xi)$ acting respectively on $\mathcal{G}_U$ and $\mathcal{G}_V$.

The advantage of the use of this representation is clear in the case when the operator $\psi(x)$ appears in the interaction in the combination $\psi(x) + C\,\overline{\psi}(x) = \sqrt{2}\;U(x)$, as in this case the field V is frozen, only transitions involving the U quantities occurring. In particular, this representation will be used later to prove that such a theory (Furry's projection theory) is isomorphic to Majorana's theory.

In order to make clear the relation between this representation and the particle-antiparticle representation (1) it is convenient to write the expressions (III-103) and (III-104) in momentum-spin space and take its hermitian conjugate; we find:

$$\psi(\xi)^{\dagger} = \frac{1}{\sqrt{2}}\left[U(\xi)^{\dagger} + V(\ )^{\dagger}\right]\ ;\ \psi'(\xi)^{\dagger} = \frac{1}{\sqrt{2}}\left[U(\xi)^{\dagger} - V(\xi)^{\dagger}\right] \quad \text{(III-113)}$$

$$U(\ )^{\dagger} = \frac{1}{\sqrt{2}}\left[\psi(\xi)^{\dagger} + \psi'(\xi)^{\dagger}\right];\quad V(\xi)^{\dagger} = \frac{1}{\sqrt{2}}\left[\psi(\xi)^{\dagger} - \psi'(\xi)^{\dagger}\right] \quad \text{(III-114)}$$

Now the wave functions for one particle or one antiparticle with a momentum-spin coordinate ξ are given, respectively, by:

$$\Psi(\xi) = \psi(\xi)^{\dagger}\Omega_0\ ;\ \Psi'(\xi) = \psi'(\xi)^{\dagger}\Omega_0\,. \quad \text{(III-115)}$$

Those for an U particle or a V particle are given by:

$$\Psi_U(\xi) = U(\xi)^{\dagger}\Omega_0;\quad \Psi_V(\xi) = V(\xi)^{\dagger}\Omega_0 \quad \text{(III-116)}$$

Thus, in view of (III-113) the wave function of a particle or an antiparticle are, in the U-V representation given by:

$$\Psi(\xi) = \frac{1}{\sqrt{2}}\left[\Psi_U(\xi) + \Psi_V(\xi)\right]\ ;\ \Psi'(\xi) = \frac{1}{\sqrt{2}}\left[\Psi_U(\xi) - \Psi_V(\xi)\right] \quad \text{(III-117)}$$

Thus we see that in the U-V representation both the negaton and positon states (for instance) are a mixture of U and V particle states.

4) Expansion into U, V'' fields.

We shall not go into the analysis of the resulting representation as it is uninteresting for our purposes.

5) Expansion in Y, Z fields (only if m = 0).

As seen in sec. C, if the mass of the field $\psi(x)$ vanishes, then the following quantities:

$$Y(x) = \frac{1}{\sqrt{2}}\left(\psi(x) + \gamma_5\, C\, \overline{\psi}(x)\right) \quad \text{(III-118)}$$

$$Z(x) = \frac{1}{\sqrt{2}}\left(\psi(x) - \gamma_5\, C\, \overline{\psi}(x)\right) \quad \text{(III-119)}$$

satisfy the same Dirac equation as $\psi(x)$. Thus we can take for our half fields the positive energy parts of Y(x) and Z(x), as they have the appropriate commutation relations. Expanding them in terms of $Y(\xi)$ and $Z(\xi)$, respectively, by 2) we find the necessary commutation relations for the application of the general factorization treatment:

$$\{Y(\xi),\ Y(\xi')^{\dagger}\} = \{Z(\xi),\ Z(\xi')^{\dagger}\} = \delta(\xi-\xi'),\quad \text{(III-120)}$$

all the other anticommutators vanishing. Thus introducing the independent half fields $\phi_y(\xi)$ and $\phi_z(\xi)$ we get the factorization:

$$\mathcal{G} = \mathcal{G}_Y \times \mathcal{G}_z \ ;\ \mathcal{H} = \mathcal{H}_Y \times \mathcal{H}_z \qquad \text{(III-121)}$$

The expression of the number of Y and Z particles, as the total energy momentum vector have the similar form to those of the previous cases, in terms of $Y(\xi)$ and $Z(\xi)$.

It is interesting to observe here that in opposition to the behaviour of the U,V fields which have the commutation relations:

$$\{U_\alpha(x),\ \overline{U}_\beta(x')\} = \{V_\alpha(x),\ \overline{V}_\beta(x')\} = \frac{1}{i} S_{\alpha\beta}(x-x') \quad \text{(III-122a)}$$

$$\{U_\alpha(x),\ U_\beta(x')\} = -\{V_\alpha(x),\ V_\beta(x')\} = -\frac{1}{i} S_{\alpha\gamma}(x-x')\ C_{\gamma\beta} \quad \text{(III-122b)}$$

$$\{U_\alpha(x),\ V_\beta(x')\} = \{U_\alpha(x),\ \overline{V}_\beta(x')\} = 0 \qquad \text{(III-122c)}$$

the commutation relations of the Y,Z fields are:

$$\{Y_\alpha(x),\ \overline{Y}_\beta(x')\} = \{Z_\alpha(x),\ \overline{Z}_\beta(x')\} = \frac{1}{i} S_{\alpha\beta}(x-x') \quad (m=0)\quad \text{(III-123a)}$$

$$\{Y_\alpha(x),\ Y_\beta(x')\} = \{Z_\alpha(x),\ Z_\beta(x')\} = \{Y_\alpha(x),\ \overline{Z}_\beta(x')\} = 0 \quad \text{(III-123b)}$$

$$\{Y_\alpha(x),\ Z_\beta(x')\} = -\frac{1}{i}\left[S(x-x')\gamma_5\ C\right]_{\alpha\beta} (m=0)\ . \quad \text{(III-123c)}$$

The commutation relations ((III-83) are interesting as they show the fundamental reason why Fireman's theory does not lead to double β decay with no neutrinos (if m = 0). This is because such a theory, where the interaction is of the type:

$$\mathcal{H} = g\,\overline{\Psi}_P \Psi_N \overline{\Psi}_e Y_\nu + \text{h.c.}, \qquad \text{(III-124)}$$

Y_ν being defined by (III-118), is isomorphic to the usual type of interaction

$$\mathcal{H} = g\,\overline{\Psi}_P \Psi_N \overline{\Psi}_e \Psi_\nu + h.c. \qquad \text{(III-125)}$$

which do not lead to such type of double β decay. This isomorphism results from the fact that the anticommutators of the quantities $Y_\nu(x)$ and $\overline{Y}_\nu(x)$ appearing in (III-124) are the same ones as for $\Psi'_\nu(x)$ and $\overline{\Psi}_\nu(x)$ which appear in (III-125) (compare (III-123 a,b).

6) Expansion in Y (particles) and Y' (antiparticles).

This case is again uninteresting and thus will not be analysed.

E) Summary of results.

The main subject of this part is the analysis of the underlying Hilbert space of wave functions for a Dirac or a Majorana field.

As a preliminary step, after some general considerations in sec. A., we analyse in sec. B the possible relativistic invariant decomposition of a Dirac field into two parts. We find that there are two such types of decomposition, say, those which are made with the well-known Schrödinger projection operators and Furry projection operators, respectively. In the case of zero mass a third type of invariant decomposition is possible, which does not correspond to a projection operation, however.

In sec. C we use these types of decomposition in order to express the total field $\Psi(x)$ in terms of two anticommuting half fields (positive energy fields). Here, there are three cases in which the half fields are the positive energy parts of respectively:

1) $\Psi(x)$ and $\Psi'(x) = C\,\overline{\Psi}(x)$

3) $U(x) = \frac{1}{2}\left[\Psi(x) + C\,\overline{\Psi}(x)\right]$; $V(x) = \frac{1}{2}\left[\Psi(x) - C\,\overline{\Psi}(x)\right]$

5) $Y(x) = \frac{1}{2}\left[\Psi(x) + \gamma_5\, C\,\overline{\Psi}(x)\right]$; $Z(x) = \frac{1}{2}\left[\Psi(x) - \gamma_5\, C\,\overline{\Psi}(x)\right]$,

the last one being possible only in the case of zero mass. The numbers 1), 3),

5) by which we designate those decompositions correspond to the order in which we consider them in that analysis. The other three possibilities, designated by 2), 4) and 6), are obtained from 1), 3) and 5) when we use for the second half field the positive energy part of the quantities $\gamma_5\psi'$, γ_5 V and γ_5 Z, respectively.

In sec. D., the analysis of the structure of the underlying Hilbert space is made by factorizing it as direct products of several subspaces, each one associated to one of the half fields appearing in the theory. These subspaces are isomorphic to each other. Several representations of the theory are obtained according to which type of decomposition we use for every Dirac field. These representations are all equivalent as they are related by unitary transformations. Some of them, say those associated with the decompositions 2), 4) and 6), are equivalent to those obtained from the decompositions 1), 3) and 5), respectively. However, they may be useful for practical computations in some cases. As an example the result that a theory of the type used by Fireman leads to no double β decay for zero neutrino mass, is proved. Also Fireman's result that the terms proportional to the mass of the neutrino in the transition probability for processes in which neutrinos are absorbed or emitted is proved with the use of one of these representations.

In the case when one of the particles described by the theory is a Majorana particle we show that only one subspace (instead of two as in the case of Dirac fields) is to be associated with this field in the factorization of the Hilbert space. This corresponds to the fact that there are no antiparticles for Majorana fields.

PART IV

THEORIES OF NEUTRAL PARTICLES WITH SPIN $\frac{1}{2}$
FERMI INTERACTIONS.

A) Neutral particles and self charge conjugation.

Before going into the analysis of the possible theories of the neutrino it is convenient to make some considerations about the conditions that could be imposed on a given field if it describes neutral particles. Some results already discussed are reproduced for the sake of clarity, the notation being the same as before.

In the case of a Boson charged field (consider of spin zero, for simplicity) the charge conjugate of the wave operator $A(x)$ is just its hermitian conjugate:

$$A'(x) = A^{\dagger}(x) \tag{IV-1}$$

The current density operator is then given by:

$$j_\mu(x) = \frac{ie}{2}\left[A^{\dagger}\frac{\partial A}{\partial x^\mu} - A\frac{\partial A^{\dagger}}{\partial x^\mu}\right] = \frac{ie}{2}\left[A^{\dagger}\frac{\partial A}{\partial x^\mu} - A'^{\dagger}\frac{\partial A'}{\partial x^\mu}\right] \tag{IV-2}$$

In order to avoid difficulties with negative energies we put, as usual:

$$A(x) = A_+(x) + A_-(x) = A_+(x) = (A'_+(x))^{\dagger} \tag{IV-3a}$$

$$A'(x) = A'_+(x) + A'_-(x) = A'_+(x) = (A_+(x))^{\dagger} \tag{IV-3b}$$

where the indices + and − indicate the positive and negative energy parts of the operator[2]; the vacuum wave function Ω_0 is defined by:

$$A_+(x)\,\Omega_0 = A'_+(x)\,\Omega_0 = 0 \tag{IV-4}$$

In the case of a Fermion the charge conjugation is equivalent to hermitian conjugation only in Majorana's representation[26] of Dirac equation. In a general representation the charge conjugated field ψ' is given by:

$$\psi'(x) = C\,\overline{\psi}(x) \tag{IV-5}$$

The current density operator is in this case:

$$j^{\mu}(x) = -\frac{ie}{2}\left[\overline{\psi}(x)\gamma^{\mu}\psi(x) - \psi(x)\gamma^{\mu}_{T}\overline{\psi}(x)\right] =$$

$$\frac{ie}{2}\left[\overline{\psi}\gamma^{\mu}\psi - \psi^{c}\gamma^{\mu}\psi^{c}\right] \qquad \text{(IV-6)}$$

In this case there are two ways of avoiding the difficulties with negative energy states:

a) Using the fact that a Fermion obeys Pauli's exclusion principle we define the vacuum as the state in which all negative energy states are occupied and the positive energy ones unoccupied, say, if we decompose the Dirac field ψ into its positive and negative energy parts:

$$\psi(x) = \psi_{+}(x) + \psi_{-}(x) \quad , \qquad \text{(IV-7)}$$

we impose:

$$\psi_{+}(x)\Omega_{0} = \psi_{-}^{\dagger}(x)\Omega_{0} = 0 \qquad \text{(IV-8)}$$

This method is somewhat cumbersome as the description of the states in configuration space becomes complicated, especially when there are particles of different charge. Moreover it seems unsatisfactory to make use of the exclusion principle in this case when the similar difficulties were solved by a more simple procedure in the case of a Boson field.

b) Similarly to the case of Boson fields we express ψ_{-} in (IV-6) in terms of the charge conjugate of the field ψ:

$$\psi_{-}(x) = C\,\overline{\psi}_{+}^{T}(x) \qquad \text{(IV-9)}$$

Then it results from (IV-5):

$$\psi^{\dagger}(x) = \psi_{+}^{\dagger}(x) + \psi_{-}^{\dagger}(x) = \psi_{+}^{\dagger}(x) + C\,\overline{\psi}_{+}(x) = C\,\overline{\psi}(x) \qquad \text{(IV-10)}$$

In this case the vacuum wave functions are characterized by:

$$\psi_{+}\Omega_{0} = \psi_{+}^{\dagger}\Omega_{0} = 0 \qquad \text{(IV-11)}$$

Now a very reasonable condition for a wave operator to represent a neutral particle is that it is to be self charge conjugate:

$$A' = A^{\dagger} = A \qquad \text{(IV-12a)}$$

$$\psi' = C\overline{\psi} = \psi \qquad \text{(IV-12b)}$$

as then the current density (IV-2) and (IV-6) would vanish identically. This would be very satisfactory as in this case it would be impossible for these fields to interact with the electromagnetic potential in the way corresponding to the existence of a charge. This assumption is the one usually adopted in the case of neutral meson fields (of integer spin). In this case it is also usual to impose that there are no antiparticles, or that instead of (IV-8a) we have:

$$A = A_+ + A_- = A_+ + A_+^{\dagger} = A' \qquad \text{(IV-13)}$$

The equivalent of this for Fermions is Majorana's condition[26), 27)]:

$$\psi = \psi_+ + \psi_- = \psi_+ + C\,\overline{\psi}_+ = \psi' \qquad \text{(IV-14)}$$

Nevertheless we cannot assume (IV-14) as universally valid for neutral particles of $\frac{1}{2}$ spin because we would face then a twofold difficulty in the case of neutrons:

First, such a neutral particle could not have an anomalous magnetic moment as the quantity:

$$\overline{\psi}_N \gamma^{\mu}\gamma^{\nu}\psi_N - \overline{\psi}_N' \gamma^{\mu}\gamma^{\nu}\psi_N' \qquad \text{(IV-15)}$$

would vanish as a consequence of (IV-14). This is in contradiction to the fact that the neutron has a magnetic moment.

Second, ψ_N would then include both creation and annihilation operators for the same particle, in view of (IV-14) and thus, both the Fermi interaction and the π meson interaction would lead to an instability of the complex nuclei.

Of course there is the possibility that the neutron should be a complex particle formed, for instance, by a particle of spin $\frac{1}{2}$ bound to a particle of integer spin and opposite charge; in this case we would impose conditions (IV-13) and (IV-14) only for the elementary neutral particles. However there is no indication that a theory of the neutron with this feature can be worked out in a satisfactory way.

We shall thus maintain the usual assumption that the neutron is an elementary particle and then, as now the conditions (IV-13) and (IV-14) cannot be anymore universal, there is no reason why they should be imposed for other neutral particles. In other words, we are free either to impose these self charge conjugation conditions or to assume that they do not hold but we have to exclude for neutral particles interactions with the electromagnetic potential of the form:

$$A_\mu(x) \quad j^\mu(x)$$

This will lead, for particles of spin $\frac{1}{2}$ to several possible theories which we shall analyse in sec. 8. Now it is convenient to make a rapid analysis of the case of Boson fields in order to see if in the case when (IV-12) is not imposed (A is not hermitian) we can be led to a theory essentially different from the usually considered for Boson neutral fields (in particular for the usual meson theories).

First we consider a case in which it would make a difference to assume that the hermiticity condition (IV-12) did not hold:

In order to consider a concrete case let us assume that π and μ mesons, as well as the neutral particle (μ_o meson) emitted in the π-μ decay have spin zero, as it is consistent with the experimental results up to now.[35] The interaction which would lead to the observed decay:

$$\pi \longrightarrow \mu + \mu_o \qquad \text{(IV-16)}$$

35). J. Tiomno, Phys. Rev. 76, 856, 1949.

is of the form:

$$A_\pi \, A^+_\mu \, A_{\mu_0} \; + \; A^+_\pi \, A_{\mu_0} \, A_\mu \qquad \text{(IV-17a)}$$

if A_{μ_0} is hermitian; otherwise it would be:

$$A_\pi \, A^+_\mu \, A_{\mu_0} \; + \; A^+_\pi \, A^+_{\mu_0} \, A_\mu \qquad \text{(IV-17b)}$$

The difference between these cases is that in case (IV-17a) we have only one type of μ_0 meson (particle = antiparticle, as we impose also condition (IV-13) for this field), and together with (IV-16) we would have also the possible inverse process:

$$\pi + \mu_0 \longrightarrow \mu \qquad \text{(IV-18)}$$

In case (IV-17b) there would exist two types of μ mesons: particles μ_0 (corresponding to the half field A_+) and antiparticles μ^+_0 (corresponding to the half field A^+_+), and thus, although (IV-18) still holds we have, instead of (IV-16)

$$\pi \longrightarrow \mu + \mu^+_0 \qquad \text{(IV-19)}$$

Now, although no difference would exist for the probabilities of first order processes as calculated in both theories, different results would be reached for higher order processes (similar to those between Dirac and Majorana types of neutrino, as in what refers to double β -decay).

Other similar situations are provided by the interactions which lead to the processes:

$$P + \mu^- \longrightarrow N + \mu_0 \qquad \text{(IV-20)}$$

$$\mu^+ \longrightarrow e^+ + U + \mu_0 \, , \qquad \text{(IV-21)}$$

assuming again spin zero for μ and μ_0 mesons. Here again it would make a difference for some second order processes to impose or not condition (IV-12) for the neutral meson wave operator. Nevertheless here as in the first example, in view of 1) the weakness of the interactions, 2) the fact that π and μ mesons are unstable particles and 3) the relatively high mass of the π meson (in relation to nuclear binding energies) it seems impossible to find a type of process, a kind of counterpart of the double β decay, which would allow a decision among the two possible types of theory. Anyway this type of consideration here is purely speculative as we do not know if the spin of μ and μ_o mesons is zero or if it is $\frac{1}{2}$, the neutral particle (which we called μ_o) being eventually a neutrino.

The intention of the preceding considerations was to point out that in some cases we will be led to different theories for neutral Boson fields if the condition of self charge conjugation is not imposed, although the differences will appear only in higher order processes and may be difficult to detect experimentally.

Now we consider the nuclear interaction of neutral π mesons (π_o meson). In this case, as indicated by the experimental evidence, the neutral meson field $A(x)$ appears linearly in the interaction hamiltonian, coupled with bilinear expression of the type $\overline{\psi}_P \psi_P$ or $\overline{\psi}_N \psi_N$ which is <u>hermitian</u>. Thus, as the interaction Hamiltonian has to be hermitian itself it is seen that it will include linearly the <u>hermitian</u> field:

$$B = A + A^{\dagger} = B^{\dagger} \qquad \text{(IV-22)}$$

Now in a way analogous to the one which we have analysed in detail in the case of $\frac{1}{2}$ spin particles the A field can be analysed either in terms of particle and antiparticle fields: A_+ and $A_+^{\dagger}$ or in terms of B_+ and C_+ fields, B being defined by (IV-22) and C by:

$$C = -i\,(A - A^{\dagger}) = C^{\dagger} \qquad \text{(IV-23)}$$

and the two methods should lead to the same results as we have no physical means of distinguishing between particle and antiparticle, or between B and C fields for neutral particles; as a consequence the theory will be isomorphic to the usual type of neutral meson theory in which the field is self charge conjugate and expanded as in (IV-13).

B) Theories of the neutrino and β -decay.

In this section we shall analyse the possible theories of the neutrino, in what concerns to the nature of the neutrino field and the way in which it appears in the interaction. As we have especially in mind the application of these results to the analysis of the double β -decay we shall consider only the Fermi type of interaction. The extension of the analysis to other types of interactions, say with mesons is immediate.

Also, in view of the fact that we were unable to find non-local theories (in the restricted sense analysed in Part I), we shall restrict ourselves to local field theories. For convenience, we consider the theory as formulated in interaction representation.

We shall consider all the half spin particles but the neutrino as described by Dirac fields, and classify the possible theories according to the behaviour of the neutrino field into:

I. "Two neutrino theories", if the neutrino is a Dirac particle and no projection operator appears in the interaction hamiltonian.

II. "Projection theories" if the neutrino is a Dirac particle but only a projection of the neutrino field operator appears in the interaction.

III. "Reduced or one-neutrino theories" if the neutrino field is a reduced field. We shall also call this type of theory "Majorana" theory as the only type of local reduced field is the Majorana field. In this case we have thus only one type of neutrino particle.

It will result from the following analysis that the only type of (local) projection theories are those isomorphic to the Majorana theory.

We shall restrict the present analysis to the case of Fermi interaction (with no derivatives). We shall also assume that in every case the phases for the improper Lorentz transformations of the several fields and in special of the neutrino field are chosen consistently with the type of interaction used, in the sense of the analysis in Part II, ... D.

I. Two neutrino theories.

We consider first the possible types of simple theories, in which the heavy particles (nucleons) and light particle (electron, neutrino) field operators are separat... combined in a quantity of integer variance. Also in this case ψ_ν and $\overline{\psi}_\nu$ should not appear in the same term. We thus classify the possible interactions into four categories, according to the behaviour of the neutrino field and in each case we consider the ... usual types of interaction:

a) Interaction of the type:

$$\overline{\psi}_P \psi_N \quad \overline{\psi}_e \psi_\nu$$

where the positive energy part of the operators ψ_P, ψ_N, ψ_e and ψ_ν are absorption operators respectively for proton (+), neutron, electron (-) and neutrino particles. The negative energy parts correspond to emission of the corresponding antiparticles.

We have thus the well-known five types of simple interactions:

Ia) Scalar: $\overline{\psi}_P \psi_N \cdot \overline{\psi}_e \psi_\nu$ + h.c.

IIa) Vector: $\overline{\psi}_P \gamma^\mu \psi_N \cdot \overline{\psi}_e \gamma_\mu \psi_\nu$ + h.c.

IIIa) Tensor: $\overline{\psi}_P \gamma^{\mu\lambda} \psi_N \cdot \overline{\psi}_e \gamma_{\mu\lambda} \psi_\nu$ + h.c.

IVa) Pseudovector: $\overline{\psi}_P \gamma_5 \gamma^\mu \psi_N \cdot \overline{\psi}_e \gamma_5 \gamma_\mu \psi_\nu$ + h.c.

Va) Pseudoscalar: $\overline{\psi}_P \cdot \gamma_5 \psi_N \overline{\psi}_e \gamma_5 \psi_\nu$ + h.c.

where $\gamma^{\mu\lambda} = \frac{1}{2}(\gamma^\mu \gamma^\lambda - \gamma^\lambda \gamma^\mu)$.

This type of Fermi interaction, proposed by Konopinski and Uhlenbeck[36] is especially convenient in the type of formulation of the theory in which the neutrino and electron are treated as two different states of isotopic spin of the same particle, in the same way as the neutron and proton.

In this type of interaction an antineutrino is emitted in the decay of a neutron, together with an electron and a neutrino is emitted (or absorbed) together with a positon (or electron). Thus if we consider the proton (+), neutrons, electrons (-) and neutrino as "particles" we can say that there is "conservation of particles" if we count the number of antiparticles as negative. For simplicity of nomenclature we generalize this concept and we shall call "theories with conservation of particles" to the theories with the following characteristic:

> "If a given process occurs in this theory in which a given particle is emitted (or absorbed) then the anti-particle cannot be emitted (or absorbed), instead of the particle, in the same process."

Surely enough, the only type of theories without conservation of particles consistent with the experimental results (nuclear stability and conservation of charge) are the ones in which a neutrino, or antineutrino, can be emitted, or absorbed, indifferently, in a given process.

b) Interactions of the type:

$$\overline{\Psi}_P \Psi_N \overline{\Psi}_e C \overline{\Psi}_\nu = \overline{\Psi}_P \Psi_N \overline{\Psi}_e \Psi_\nu'$$

Here the five types of interactions are:

Ib)	Scalar:	$\overline{\Psi}_P \Psi_N \overline{\Psi}_e C \overline{\Psi}_\nu$ + h.c.
IIb)	Vector:	$\overline{\Psi}_P \gamma^\mu \Psi_N \overline{\Psi}_e \gamma_\mu C \overline{\Psi}_\nu$ + h.c.
IIIb)	Tensor:	$\overline{\Psi}_P \gamma^{\lambda\mu} \Psi_N \overline{\Psi}_e \gamma_{\lambda\mu} C \overline{\Psi}_\nu$ + h.c.

36). E. V. Konopinski and G. E. Uhlenbeck, Phys. Rev. 48, 7, 1935.

IVb) Pseudovector: $\overline{\psi}_P \gamma_5 \gamma^\mu \psi_N \overline{\psi}_e \gamma_5 \gamma_\mu C \overline{\psi}_\nu$ + h.c.

Vb) Pseudoscalar: $\overline{\psi}_P \gamma_5 \psi_N \overline{\psi}_e \gamma_5 C \overline{\psi}_\nu$ + h.c.

Here a neutrino is emitted together with an electron and an antineutrino together with a positon. In this type of theory, as well as in all the other of "simple" interaction there is conservation of particles in the sense referred to above.

c) Interactions of the type:

$$\overline{\psi}_P \psi_N \overline{\psi}_e \gamma_5 C \overline{\psi}_\nu = \overline{\psi}_P \psi_N \overline{\psi}_e \psi_\nu''$$

The simple interactions are here:

Ic) Scalar $\overline{\psi}_P \psi_N \overline{\psi}_e \gamma_5 C \overline{\psi}_\nu$ + h.c.

IIc) Vector: $\overline{\psi}_P \gamma^\mu \psi_N \overline{\psi}_e \gamma_5 \gamma_\mu C \overline{\psi}_\nu$ + h.c.

IIIc) Tensor: $\overline{\psi}_P \gamma^{\mu\lambda} \psi_N \overline{\psi}_e \gamma_5 \gamma_{\mu\lambda} C \overline{\psi}_\nu$ + h.c.

IVc) Pseudovector: $\overline{\psi}_P \gamma_5 \gamma^\mu \psi_N \overline{\psi}_e \gamma_\mu C \overline{\psi}_\nu$ + h.c.

Vc) Pseudoscalar: $\overline{\psi}_P \gamma_5 \psi_N \overline{\psi}_e C \overline{\psi}_\nu$ + h.c.

As in cases b) a neutrino (or antineutrino) is emitted together with an electron (or positon). One should observe that the vector interaction IIc) is the one used by Fermi in the original formulation of the β-decay theory[37].

d) Interactions of the type:

$$\overline{\psi}_P \psi_N \overline{\psi}_e \gamma_5 \psi_\nu = - \overline{\psi}_P \psi_N \overline{\psi}_e C \overline{\psi}'' .$$

37) E. Fermi, Zeits. f. Phys., 88, 161, 1934.

with the five simple interactions:

Id) Scalar: $\bar{\psi}_P \psi_N \quad \bar{\psi}_e \gamma_5 \psi_\nu$ + h.c.

IId) Vector: $\bar{\psi}_P \gamma^\mu \psi_N \, \bar{\psi}_e \gamma_\mu \gamma_5 \psi_\nu$ + h.c.

IIId) Tensor: $\bar{\psi}_P \gamma^{\mu\lambda} \psi_N \, \bar{\psi}_e \gamma_{\mu\lambda} \gamma_5 \psi_\nu$ + h.c.

IVd) Pseudovector: $\bar{\psi}_P \gamma_5 \gamma^\mu \psi_N \, \bar{\psi}_e \gamma_\mu \psi_\nu$ + h.c.

Vd) Pseudoscalar: $\bar{\psi}_P \gamma_5 \psi_N \quad \bar{\psi}_e \psi_\nu$ + h.c.

These theories have the same behaviour as those of group a) in what refers to the type of neutrino (particle or antiparticle) emitted or absorbed together with electron and positon.

It should be observed that once a convenient choice is made of the phase factors for the transformation of the several fields under the improper Lorentz group then only theories of one of the following bracketed groups of theories are possible: (a, b), (a, c), (d, b) or (d, c). For instance, if a choice is made such that the expressions listed in group a) are invariants then only those of group b) or c) will be invariant according to which of C . $\bar{\psi}_\nu$ or γ_5 C $\bar{\psi}_\nu$, respectively, transforms as ψ_ν (see Part II, sec. D).

Before passing to the consideration of the mixed theories a few observations should be made on the behaviour of simple theories of the same name (scalar, vector, etc.) belonging to different categories.

1) Corresponding theories of group a) and b) are isomorphic; also corresponding theories of group c) and d) are isomorphic.

This property is usually expressed by saying that there is a symmetry between neutrino and antineutrino (or, in general, between particle and antiparticle).

It results from the following facts already analysed in Part III.

α) These theories go into each other by the transformation:

$$\psi_\nu \longrightarrow \psi'_\nu = C\,\overline{\psi}_\nu \qquad \text{(IV-24)}$$

β) The predictions of a given theory are the same if the factorisation of the underlying Hilbert space $\mathcal{G}$ and of the algebra $\mathcal{A}$ corresponding to the neutrino field, is done respectively as:

$$\mathcal{G} = \mathcal{G}_p \times \mathcal{G}_a \quad ; \mathcal{A} = \mathcal{A}_p \times \mathcal{A}_a \qquad \text{(IV-25)}$$

or as:

$$\mathcal{G} = \mathcal{G}_a \times \mathcal{G}_p \; ; \; \mathcal{A} = \mathcal{A}_a \times \mathcal{A}_p, \qquad \text{(IV-26)}$$

in view of the isomorphism of $\mathcal{G}_a$ on $\mathcal{G}_p$ or of $\mathcal{A}_a$ on $\mathcal{A}_p$. Now if we use the factorisation (IV-25) for one of the cases (say a) and (IV-26) for the other (say b), then the transformation (IV-24) will bring one into the other.

γ) Both $\psi(x)$ and $\psi'(x)$ satisfy the <u>same</u> Dirac equation and thus when summing over the two spin possibilities the same Schrödinger - Casimir projection operator will appear in both cases.

2) Corresponding theories of the groups a) and c) or of b) and d) are isomorphic if the mass of the neutrino vanishes ($m = 0$). If $m \neq 0$ the matrix elements for any process with real emission of neutrinos, as obtained from the two theories to be compared, differ by the sign of the terms in m_ν (coming from the Schrödinger-Casimir projection operators corresponding to the several real emissions or absorptions of neutrinos):

$$m_\nu \longrightarrow -m_\nu \qquad \text{(IV-27)}$$

This affirmation is made obvious by the following observations:

α) The theories referred to will go into each other by the transformation:

$$\Psi_\nu \longrightarrow \Psi''_\nu = \gamma_5 C \overline{\Psi}_\nu \qquad \text{(IV-28)}$$

β) For the factorisation of the underlying Hilbert space $\mathcal{G}$ and the algebra of operators $\mathcal{A}$ we have:

$$\mathcal{G} = \mathcal{G}_p \times \mathcal{G}_a \; ; \; \mathcal{A} = \mathcal{A}_p \times \mathcal{A}_a \qquad \text{(IV-29)}$$

in case a) or b) and

$$\mathcal{G}'' = \mathcal{G}_p \times \mathcal{G}_c \; ; \mathcal{A}'' = \mathcal{A}_p \times \mathcal{A}_c \quad \text{(IV-30)}$$

in case c) or d).

The isomorphism of $\mathcal{G}_a$ on $\mathcal{G}_c$ and of $\mathcal{A}_a$ on $\mathcal{A}_c$ assures the isomorphism of $\mathcal{G}$ on $\mathcal{G}''$ and of $\mathcal{A}$ on $\mathcal{A}''$.

γ) $\psi''(x) = \gamma_5 C \overline{\psi}(x)$ satisfies a Dirac equation with the opposite sign of the mass term in relation to that for $\Psi(x)$. Thus in summation over the two spin directions of the neutrino states a Schrödinger-Casimir projection operator will appear (for every real emission or absorption) which has different sign in the two theories being compared. Thus there will be no difference if $m = 0$. The contributions from virtual transitions are identical even if $m \neq 0$ as no projection operator will appear in view of the summation over both positive and negative energy states for the intermediate neutrino considered.

As an example we observe that the angular correlation factor for simple β-decay, which is given in the scalar theory by:

$$\alpha = 1 - \frac{\vec{P}_\nu \cdot \vec{P}_e \, c^2}{E_\nu \, E_e} + \frac{m_\nu \, m_e \, c^4}{E_\nu \, E_e} \qquad \text{(IV-31)}$$

in cases a) b), is given by:

$$\alpha = 1 - \frac{\vec{P} \cdot \vec{P} \, c^2}{E_\nu \, E_e} - \frac{m_\nu m_e \, c^4}{E_\nu E_e} \qquad \text{(IV-32)}$$

in the cases of theories c), d).

An experimental determination of the sign of the last term in the angular correlation factor would be of help for a decision among the cases of theories of the types a), b), or c), d). The present experimental results are, however, insufficient for this differentiation in view of the smallness of the neutrino mass.

As a final observation we notice that the analysed isomorphism between a given theory and the one obtained from it by the transformation:

$$\psi_\nu \longrightarrow C\ \overline{\psi}_\nu \qquad \text{(IV-24)}$$

does not lead to the result that a theory obtained from anyone of those referred to in this section by a substitution:

$$\psi_\nu \longrightarrow \psi^{(1)} = \frac{1}{1+\lambda}\left(\psi_\nu + C\ \overline{\psi}_\nu\right), \qquad \text{(IV-33)}$$

where λ is a constant, is isomorphic to the original one. This is made clear by the fact that, although $\psi'(x) = C\ \overline{\psi}(x)$ has the same anticommutation relations as $\psi(x)$, the quantity $\psi^{(1)}(x)$ has different anticommutation relations, say:

$$\left\{\psi^{(1)}_\alpha(x),\ \psi^{(1)}_\beta(x')\right\} = \frac{i\ \lambda}{(1+\lambda)^2}\left(S\ (x-x')\ C\right)_{\alpha\beta} \qquad \text{(IV-34)}$$

$$\left\{\psi^{(1)}_\alpha(x),\ \overline{\psi}^{(1)}_\beta(x')\right\} = \frac{1}{i}\ \frac{1+|\lambda|^2}{(1+\lambda)^2}\ S_{\alpha\beta}(x-x') \qquad \text{(IV-35)}$$

The fact that the first anticommutator (IV-34) does not vanish for finite λ will lead to the possibility of double β-decay in the new theory, although this was not possible in the old one.

However, in the case where $m = 0$, the substitution:

$$\psi \longrightarrow \psi^{(2)} = \frac{1}{\sqrt{1+|\lambda|^2}}\left(\psi + \gamma_5\ C\ \overline{\psi}\right) \qquad \text{(IV-36)}$$

will lead to a theory isomorphic to the original one as we have then for the commutation relations of $\psi^{(2)}(x)$:

$$\{\psi_{\alpha}^{(2)}(x),\ \psi_{\beta}^{(2)}(x')\} = 0 \qquad \text{(IV-37)}$$

$$\{\psi_{\alpha}^{(2)}(x),\ \overline{\psi}_{\beta}^{(2)}(x)\} = \frac{1}{i}\, S_{\alpha\beta}(x-x') \qquad \text{(IV-38)}$$

This result is a generalization of the one proved before, say that Fireman's theory (which corresponds to take $\lambda = 1$) is isomorphic to a theory with conservation of particles. Here, as in Fireman's theory, the vanishing of the anticommutator (IV-37) leads to the result that no double β-decay with no neutrino occurs[38].

Mixed theories:

We shall classify the mixed theories into two groups, according to if there is or there is not conservation of particles.

1) Theories with conservation of particles.

We could characterise these theories in a general way by saying that the interaction hamiltonian is invariant under a phase transformation

$$\psi(r) \longrightarrow f\,\psi(r) \qquad f^* f = 1$$

f being an arbitrary phase factor, the same for all particles or, at most, the complex conjugate for the neutrino field:

$$\psi_{\nu}' \longrightarrow f^*\, \psi_{\nu} \quad .$$

38). B. Toushek, Zeits. f. Phys. 125, 108, 1948. In this paper the theory indicated by (I) is a mixed theory of the type (IV-36), with an interaction, say in the scalar case of the form:

$$\mathcal{H} = g\, \overline{\psi}_P\, \psi_N\, \overline{\psi}_E (\psi_{\nu} + \lambda \gamma_5\, C\, \overline{\psi}_{\nu}) \quad + \text{h.c.}$$

In order to reduce Toushek's expression (I) in page 116 to the form above we should put $Q = \beta$ (scalar theory) and $T = \gamma_5\, C\, \beta_T$, the last substitution resulting from the fact that in the Dirac's representation, used by Toushek, $\alpha_y = C$; Toushek found by direct computation that such a theory do not lead to double β-decay with no neutrinos.

In practical terms, a theory of this type can be obtained by linear combination of simple interactions belonging to the same category (a), b), c) or d)).

The reason why this type of theory should be considered for the nuclear β-decay comes from the fact that no one of the simple theories considered before give correct predictions for all the known forbidden spectra, although all of them lead to the same result (if the mass of the neutrino vanishes) for allowed spectra, result which is in good agreement with the experience.

Fierz[39] has shown that the most general interaction of this type is obtained by a linear combination, with arbitrary coefficients, of the five simple interactions of any of the four groups (a), b), c) or d)) previously considered. Practically all of these combinations, but some very special ones which will not be considered here, lead to the same allowed spectrum as the simple theories.

One should observe here that the theories obtained from group a) or c) are equivalent, respectively, to those obtained from group b) or d). Also, if the mass of the neutrino vanishes (as it nearly does) then all the four types of mixed theories are equivalent.

In this type of theory no double β-decay without neutrinos occur.

Wigner-Critchfield theory:

There is an special theory of this group, which although being a definite mixture of the scalar, pseudovector and pseudoscalar (with weights, respectively, 1, -1 and 1) which can be expressed in a very simple form and should perhaps be included among the simple theories as only one arbitrary constant, is used in it. This is the totally antisymmetric interaction pro-

39). M. Fierz, Zeits. f. Phys., 104, 553, 1937.

posed by Wigner and Critchfield[40], which is in qualitative agreement with the experimental results on simple β-decay.

The interaction hamiltonian for this theory can be expressed in the form:

$$\varepsilon_{\mu\lambda\rho\sigma}\ \Psi^{\mu}_{P'}\ \Psi_{N}\ \Psi_{e'}\ \Psi_{\nu}\ +\text{h.c.} \qquad \text{(IV-39)}$$

where the indices P', N, e' and ν refer respectively to antiproton, neutron, antielectron (positon) and neutrino, the positive energy part of the Ψ's being absorption operator for these particles. $\varepsilon_{\mu\lambda\rho\sigma}$ is the well-known antisymmetric tensor.

This interaction has the following remarkable property:

It is invariant under a reordering of the several fields appearing on it (if different spinor fields do anticommute as we have found convenient in order that we should be able to pass from interaction representation to Heisenberg representation).

This property is especially interesting as it makes the interaction (IV-39) the only one appropriate for a formulation of a general theory of Fermi interaction among all the spin $\frac{1}{2}$ particles (assuming that the μ-meson has spin $\frac{1}{2}$), on which all the spinor fields are dealt with in the same footing. This type of universal interaction, suggested by the fact that the coupling constant of a Fermi interaction which would lead to the observed μ-decay and of that leading to the μ-capture have the same order of magnitude, has been analyzed by Yang and Tiomno[22]. The predictions of a theory with a Wigner-Critchfield interaction leading to μ-decay has been analyzed by Michel[41] in what concern the energy spectrum of the emitted electron. It is in qualitative agreement with the experimental results[42], although the experimental errors are still large to permit a decision, especially in the high energy end.

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40). C. L. Critchfield and E. P. Wigner, Phys. Rev. 60, 412, 1941; C. L. Critchfield, Phys. Rev. 63, 417, 1943.

41) L. Michel, Proc.London Phys. Soc., 63A,514,1950

42) R.B. Leighton, C.D.Anderson and A.H. Seriff, Phys. Rev. 75, 1432, 1949.

2) Theories without "conservation of particles".

In this group we include those theories in which either a neutrino or an antineutrino can be emitted (or absorbed) in a given process. The immediate consequence of this fact is the possibility of a double β-decay without emission of neutrinos.

The interaction for a theory of this type is, in the most general case, given by a linear combination, with ten arbitrary constants, of:

α) Interactions of groups a) <u>and</u> b) (if $C\,\overline{\Psi_\nu}$ transforms under the improper Lorentz group as ψ_ν).

β) Interactions of groups a) and c) (if $\gamma_5 C\,\overline{\Psi_\nu}$ transforms as ψ_ν).

γ) Interactions of groups d) and b) (if $\gamma_5 C\,\overline{\Psi_\nu}$ transforms as ψ_ν).

δ) Interactions of groups d) and c) (if $C\,\overline{\Psi_\nu}$ transforms as ψ_ν).

We shall denote the theories corresponding to the cases a), b), c) and d) respectively by (a,b), (a,c), (d,b) and (d,c).

If $m = 0$ then every theory of the type (d,b) or (c,d) is isomorphic respectively, to a theory of the type (a,c) or (a,b). This is a consequence of the fact that they go into each other by the transformation:

$$\psi_\nu \longrightarrow \psi_\nu''' = \gamma_5 \psi_\nu \qquad \text{(IV-40)}$$

and by the fact that ψ_ν''' satisfies the same Dirac equation as ψ_ν if the mass of the neutrino vanishes.

If $m \neq 0$ there will be differences in the predictions of the theories being compared, coming from the fact that the mass term in the Dirac equations (and thus in the Schrödinger-Casimir projection operators) for ψ_ν and ψ_ν''' have opposite signs.

For our purpose it is a good approximation to assume $m = 0$ and thus we shall consider only theories of the type (a,b) and (a,c).

The simplest type of such theories are those which are obtained from the interactions of group a) by the substitution:

$$\psi_\nu \longrightarrow \psi_\nu + \lambda\, c\, \overline{\psi_\nu}^{\,20)} \quad ((a,b) \text{ theory }) \qquad (IV-41)$$

or, by the substitution:

$$\psi_\nu \longrightarrow \psi_\nu + \lambda \gamma_5\, c\, \overline{\psi_\nu}^{\,(38)} \quad ((a,c) \text{ theory}) \qquad (IV-42)$$

where λ is an arbitrary constant.

In the special case when $\lambda = 1$ the substitution (IV-41) leads to a Furry projection theory which will be considered in the group of projection theories (section B) and the substitution (IV-42) leads to a Fireman theory.

Computation of the double β emission will be made in Part V for the theories obtained by substitutions (IV-41). The theories resulting from the substitution (IV-42) in any of the interactions of the group a) do not lead to double β-decay with no emission of neutrinos as we have shown before. It should be pointed out that this does not mean that no theory of type (a,c) leads to double β-decay. For instance in a theory with an interaction which is a mixture of a scalar interaction Ia) with a pseudo-scalar interaction Vc) there will be such a double β-decay[38]. We shall not, however, consider these cases in the present analysis in the same way as we shall forget about the general case of an (a,b) theory, although there is no reason, in principle, to exclude them.

II. Projection theories.

If we consider any kind of two-neutrinos theory and make a substitution:

$$\psi_\nu \longrightarrow P\, \psi_\nu \qquad (IV-43)$$

where P is a projection operator we obtain a "projection theory".

Now, as seen in Part III, we have only two kinds of relativistic invariant projection operators which allow us to split the field Ψ into two projected fields Ψ_1 and Ψ_2:

$$\Psi_1 = P\,\Psi \qquad \text{(IV-44a)}$$

$$\Psi_2 = \Psi - P\Psi = P'\,\Psi \qquad \text{(IV-44b)}$$

where P and P' satisfy the condition:

$$P^2 = P \;;\; P'^2 = P' \;;\; PP' = P'P = 0 \qquad \text{(IV-45)}$$

These are the Furry projection operator:

$$P_F\Psi = \frac{1}{2}(\Psi + C\,\overline{\Psi}) \qquad \text{(IV-46)}$$

and the Schrodinger projection operator:

$$P_+\,\Psi = \Psi_+ \qquad \text{(IV-47)}$$

Thus we have only two types of projection theories:

a) Furry projection theories.

b) Schrodinger projection theories.

It is clear that if in the case of a Furry projection theory we use a representation in which the operators for the number of particles of the U and V fields (see part (III) are diagonal (for every value of the momentum and spin) then only transitions involving the U particles will occur as the V fields do not appear in the interaction. Similarly if we use, in the case of a Schrödinger projection theory the usual "particle-antiparticle" representation only transitions involving "particles" will occur. Thus, although a projection theory involves a two-neutrino field, only one of the two particles will be involved in actual transitions (if we use another representation than the "natural" ones referred to above we have to consider both types of particles. This is the case, for instance, for the Furry projection theory if we use the "particle-antiparticle" representation as will be done in Part V.

The commutation relations for the U fields,

$$U = \frac{1}{\sqrt{2}} \left(\Psi + C \overline{\Psi} \right) \qquad \text{(IV-48)}$$

are the following ones:

$$\{ U_\alpha(x), \; U_\beta(x') \} = -\frac{1}{i} S_{\alpha\delta}(x-x') \, C_{\delta\beta} \qquad \text{(IV-49a)}$$

$$\{ U_\alpha(x), \; \overline{U}_\beta(x') \} = \frac{1}{i} S_{\alpha\beta}(x-x') \qquad \text{(IV-49b)}$$

which vanish for x-x' spacelike.

Thus any Furry projection theory is a "local theory".

This is not, however, the case for any Schrödinger projection theory, in special for those obtained from any "simple" theory, as $\Psi_+(x)$ and $\overline{\Psi_+}(x')$ do not have a vanishing anticommutator for x-x' space-like. We have to exclude such cases, as the resulting theories (which we were able to formulate only in the S-matrix form in Part I) are not relativistic invariant. The same is true for the Schrödinger projection theory obtained from Fireman's mixed theory - and this eliminates the hope that one could justify Fireman's results, which came from his neglect of the antineutrino intermediate states.

In order that a Schrödinger projection theory should be relativistically invariant (and also a local theory) it is necessary that $(\Psi_\nu)_+$ should appear in the interaction combined to other neutrino quantity in such a way that this combination has vanishing commutation relations at points connected by spacelike vectors. The only such a type of invariant combination is:

$$W = \frac{1}{\sqrt{2}} \left[\Psi_+ + C \overline{\Psi_+} \right] \qquad \text{(IV-50)}$$

which has the commutation relations:

$$\{W_\alpha(x),\ W_\beta(x')\} = -\tfrac{i}{2} S_{\alpha\gamma}(x-x')C_{\gamma\beta} \qquad \text{(IV-51a)}$$

$$\{W_\alpha(x),\ \overline{W}_\beta(x')\} = \tfrac{i}{2} S_{\alpha\beta}(x-x') \qquad \text{(IV-51b)}$$

Now, as the field W (x) has the same commutation relations as U(x), it is clear that this Schrödinger projection theory will be isomorphic on a Furry projection theory. Actually both this type of Schrödinger projection and the Furry projection theory will be shown in the next section to be isomorphic to a Majorana theory.

III. Reduced or one-neutrino theories (Majorana).

As it was already seen the only type of reduced local theory is a Majorana theory, in which the neutrino field $\mathcal{U}(x)$ satisfies the self charge conjugation condition:

$$\mathcal{U}(x) = C\,\overline{\mathcal{U}}(x) \qquad \text{(IV-52)}$$

This condition, it was said before, is the equivalent of the hermiticity condition usually assumed for neutral Boson fields. Indeed in Majorana's representation of Dirac equation it takes the form:

$$\mathcal{U}(x) = \mathcal{U}^\dagger(x) \qquad \text{(IV-53)}$$

Majorana's motivation for the suggestion of this type of theory[26) was that it should be assumed as a general principle that a neutral elementary particle should be described by a self charge conjugate field, for which the antisymmetrized expression for the charge current density automatically vanishes. This principle, however, cannot be accepted as general if one assumes, as usual, that the neutron is an elementary particle, as it was discussed in the beginning of this Part.

The commutation relations for a Majorana field are:

$$\{\mathcal{U}_\alpha(x), \mathcal{U}_\beta(x')\} = -\frac{1}{i} S_{\alpha\gamma}(x-x')\, C_{\gamma\beta} \qquad \text{(IV-54a)}$$

$$\{\mathcal{U}_\alpha(x), \overline{\mathcal{U}}_\beta(x')\} = \frac{1}{i} S_{\alpha\beta}(x-x') \qquad \text{(IV-54b)}$$

The underlying Hilbert space for a Majorana field was seen in Part III to be isomorphic on $\mathcal{G}_U$ (subspace corresponding to U particles) and $\mathcal{G}_P$ ("particle" subspace).

This fact, together with the fact that the commutation relations (IV-54) for $\mathcal{U}$ field are identical to (IV-49) and IV-51) for U and W fields, respectively, shows that a Furry projection theory with the interaction expressed in terms of U-field and a Schrödinger projection theory of the type considered above (local) with the interaction expressed in terms of the W field are both isomorphic to a Majorana theory which is obtained from them by the substitution

$$U(x) \longrightarrow \mathcal{U}(x)$$

or

$$W(x) \longrightarrow \mathcal{U}(x),$$

respectively.

This result will be verified, for Furry projection theory, in the special case of double β-decay by actual computation of the transition probability in the "particle-antiparticle" representation.

It should be observed, as an argument favorable to Majorana's theory (besides Majorana's own argument) that this is the simplest of the theories which lead to double β-decay with no neutrino, as the most general Majorana interaction involves only five arbitrary constants instead of ten for a "two-neutrino theory".

PART V

APPLICATIONS TO DOUBLE β -DECAY AND ANALYSIS OF THE EXPERIMENTAL RESULTS

A. Generalities.

We call, for short, "double β-decay" a process in which a given nucleus emits two electrons and no neutrinos, with a change of two neutrons into protons. As already remarked in part IV such a process can occur, for interactions of the "simple" type, only in Majorana Theory and in a mixed theory of the type (a,b). For the usual theories with conservation of neutrinos and for mixed theories of type (a,c) with a simple interaction (Fireman's theory) only the double β decay with two neutrinos is possible. However, if we mix a theory of type a) (say, scalar) with a different theory of type c), (say, vector) double β decay may become possible. We shall exclude, for simplicity, this case. Also we will consider only the case when the initial atom cannot suffer simple β -decay because it has a smaller mass than the atom with one less neutron and one more proton. The possibility of double β -decay will arise if the atom with two neutrons less and two protons more is lighter than the initial one.

In the following sections we shall consider first the case of Majorana theories and then that of mixed (a,b) theories. It is convenient to recapitulate here the essential features of these theories and verify their normalization for the simple β -decay.

1. Majorana theory:

In this type of theory, as seen before, the neutrino is characterized by a Majorana spinor field $\mathcal{U}_{\alpha}(x)$ which satisfies the self charge conjugation condition:

$$C\overline{\mathcal{U}}(x) = \mathcal{U}(x) \qquad (V\text{-}1)$$

If $\mathcal{U}_+(x)$ is the positive energy part of $\mathcal{U}(x)$, then the total field will be expressed by:

$$\mathcal{U}(x) = \mathcal{U}_+(x) + C\,\overline{\mathcal{U}}_+(x) \qquad \text{(V-2)}$$

Relation (V-2) corresponds to the usual expression of an integer spin field describing neutral particles as:

$$A(x) = A_+(x) + A_+(x)^{\dagger} \qquad \text{(V-3)}$$

where $A_+(x)$ is the positive energy part of $A(x)$.

In the case of a scalar theory we will have the interaction hamiltonian:

$$\mathcal{H}(x) = g\,\overline{\Psi}_P(x)\,\Psi_N(x)\,\cdot\,\overline{\Psi}_e(x)\,\mathcal{U}(x) \quad + \text{ c.c.} \qquad \text{(V-4)}$$

where Ψ_P, Ψ_N and Ψ_e are the usual field operators for proton, neutron and electron (negaton), respectively.

The anticommutation relations for $\mathcal{U}_+(x)$ are:

$$\left\{ \mathcal{U}_{+,\alpha}(x)\,,\,\overline{\mathcal{U}}_{+,\beta}(x') \right\} = \frac{1}{i}\,S^{(+)}_{\alpha\beta}\,(x-x') \qquad \text{(V-5)}$$

where $S^{(+)}(x-x')$ is the positive energy part of:

$$S\ \ (x-x') = \gamma^{\mu}\ \frac{\partial}{\partial x^{\mu}}\ D(x-x') \qquad \text{(V-5a)}$$

(we take the mass of the neutrino equal to zero). Field operators corresponding to different particles anticommute.

The anticommutators of the total neutrino field will be, thus:

$$\left\{ \mathcal{U}_{\alpha}(x),\ \mathcal{U}_{\beta}(x') \right\} = -\ \frac{1}{i}\ S_{\alpha\gamma}(x-x')\ C_{\gamma\beta} \qquad \text{(V-6a)}$$

$$\left\{ \mathcal{U}_{\alpha}(x),\ \overline{\mathcal{U}}_{\beta}(x') \right\} = \frac{1}{i}\ S_{\alpha\beta}(x-x') \qquad \text{(V-6b)}$$

It should be observed that these anticommutators have twice the value of those used by Majorana[26] and Furry[27]. As a consequence the expression for the kinetic energy operator, for instance, will be:

$$H_0 = \frac{i}{2} \int \overline{\mathcal{U}}(x) \; (\vec{\alpha} \cdot \vec{\nabla}) \, \mathcal{U}(x) \; d_3 x \qquad \text{(V-7)}$$

We prefer this formulation as it brings a closer similarity with the case of integer spin neutral particles where the expression for the energy operator differs also by a factor 1/2 from those for the corresponding charged particles. Another advantage is that the result for a single β-decay process in Majorana theory will be the same as for the corresponding usual (Fermi-type) theory, with the same value of g, no normalization factor being needed, as is easy to verify.

2. Mixed theories of the type (a,b):

It is convenient to normalise the interaction here in such a way that the total transition probability for single β-decay (both with emission of neutrino or of antineutrino) be the same as for the corresponding usual theory with the same g. This result will be obtained if we write the interaction hamiltonian, say for the scalar case, as:

$$\mathcal{H}(x) = \frac{g}{\sqrt{1+|\lambda|^2}} \; \overline{\Psi}_P(x) \; \Psi_N(x) \cdot \overline{\Psi}_e(x) \; (\Psi_\nu(x) + \lambda C \overline{\Psi}_\nu(x)) \; + \text{h.c.} \qquad \text{(V-8)}$$

It is enough to consider here the case

$$|\lambda| \leq 1$$

as otherwise we would include λ in g and have instead of $\Psi_\nu + \lambda C \overline{\Psi_\nu}$, $C\overline{\Psi}_\nu + \frac{1}{\lambda}\Psi_\nu = \Psi'_\nu + \frac{1}{\lambda} C \overline{\Psi'_\nu}$ which leads to a theory isomorphic to the one where Ψ_ν appears in the combination $\Psi + \lambda' C \overline{\Psi_\nu}$ where

$$\lambda' = \frac{1}{\lambda} < 1.$$

3. Double β-decay with emission of two neutrinos.

This process has been computed by M. Goeppert-Mayer[43] for the case of usual (Fermi-type) theories in which case the half-life t is given, for

43) M. Goeppert Mayer, Phys. Rev., 48, 512, 1935.

for allowed transitions, by

$$\frac{1}{\tau} = \frac{0.69 \times 4\pi^{8} \alpha^{2} Z^{2}}{6 \times 7 \times 15 \left| \Gamma(3+2S) \right|^{2}} \left(\frac{2m\, c\rho}{\hbar} \right)^{4S} \varepsilon^{4} \; \frac{m^{9} c^{6}}{h^{13}} \; F(\varepsilon - 2) \qquad (V\text{-}9)$$

where:

$$S = 1 - \sqrt{1-\alpha^{2} Z^{2}} \; ; \quad F(x) = x^{7}\left(1 + \frac{1}{2}x + \frac{1}{9}x^{2} + \frac{1}{90}x^{3} + \frac{1}{1980}x^{4}\right), \qquad (V\text{-}9a)$$

for all the simple theories. It is convenient to refer here to the energy spectrum of the emitted electrons, for further comparison with the case of double β-decay without neutrinos. This energy spectrum is of the form: (for the case of double decay with two neutrinos)

$$F(h_s) \propto h_s^{2} \; \left[(\varepsilon - h_s)^{8} - 1 \right] . \qquad (V\text{-}10)$$

In (V-9) and V-10) ρ is the nuclear radius, h_s and ε the electron energy and nuclear change of energy, respectively, in units $m c^{2}$.

Of course, in the case of Majorana theory and of mixed theories of type (a,b) some double β-decay with emission of neutrinos will occur in competition with the double β-decay, with a probability given by expression (V-9), which will be, however, very small in relation to that for double β-decay without neutrinos. For this reason we will neglect these contributions in what follows.

B. Double β-decay without neutrinos.

Before we go into the actual computation of the probability for double β-decay it is convenient to consider the selection rules which may possibly apply in various cases. In all the possible cases of double β-decay both the initial and the final nucleus are even-even nuclei and thus both will in the ground state have, very probably, spin 0. Now, from the shell model we should expect that they will have the same parity. Thus, in general, we should have to deal with a case: $0 \longrightarrow 0$ (no). Therefore we are not necessarily justified in neglecting a term which looks formally smaller than another, as the second one may vanish in virtue of the selection rules. On the other hand, this second term could give a sizable contribution for transitions to slightly excited states.

1) Majorana theories.

We consider first the case of Majorana theories (and we shall restrict ourselves, as said before, to the cases of simple interactions). The fact that the Majorana theory would lead to double β-decay with a much larger probability than that for double β-decay with two neutrinos (which is the only possibility for the usual type of theories) was first shown by Furry[44] in a fundamental paper whose notation we follow as far as possible. However, now that a comparison with experiment is possible, we think that an approximation as drastic as the one he used; viz, assuming that only one nuclear intermediate state gives a large contribution to the transition probability, should be avoided. Using a different approximation, which seems much more reasonable, we shall obtain a result significantly different from Furry's.

a) Scalar theory:

The application of the usual perturbation treatment to the second order process in question leads to the following probability per unit time of a double β-decay in which one electron is emitted with an energy in the interval H_s to $H_s + dH_s$ (and the other electron with the energy $H_t = E_M - E_L - H_s$, where E_M and E_L are, respectively, the energies of the initial and final nuclei):

$$P(H_s)\frac{dH_s}{mc^2} = \frac{2\pi}{mc^2 \hbar} \sum_{k,k',N} \left| a^{k,k'}_{N \leftarrow M}(H_s) \right|^2 \frac{dH_s}{mc^2} \qquad \text{(V-11)}$$

where the amplitude a is given by:

44) W. H. Furry, Phys. Rev. 56, 1184, 1939.

$$a^{k,k'}_{N\leftarrow M}(H_s) =$$

$$= \frac{g^2}{(2\pi)^3}\sum_L \sum_{i,j} d_3 k \sum_{s=1}^{2}\Bigg\{\int \psi_N^*(X)\,\beta_j Q_j^{\dagger}\, e^{i\vec{k}\cdot\vec{x}_j}\,\psi_L(X)\,dX \int \psi_L^*(X')\,\beta_i\, Q_i^{\dagger}\, e^{-i\vec{k}\cdot\vec{x}'_i}\,\psi_M(X')\,dX'$$

$$\cdot\left[\frac{\left(\varphi_t^{(K)^*}(\vec{x}_j)\beta\varphi(\vec{k},s)\right)\left(\varphi^*(\vec{k},s)C\varphi_s^{(K')^*}(\vec{x}'_i)\right)}{E_M - E_L - H_s - K_k} - \frac{\left(\varphi_s^{(K')^*}(\vec{x}_j)\beta\varphi(\vec{k},s)\right)\left(\varphi^*(\vec{k},s)C\,\varphi_t^{(K)^*}(\vec{x}'_i)\right)}{E_M - E_L - H_t - K_k}\right] -$$

$$- \int \psi_N^*(X)\,\beta_i\, Q_i^{\dagger}\, e^{-i\vec{k}\cdot\vec{x}_i}\,\psi_L(X)\,dX \cdot \int \psi_L^*(X')\,\beta_j\, Q_j^{\dagger}\, e^{i\vec{k}\cdot\vec{x}'_j}\,\psi_M(X')\,dX' \cdot$$

$$\cdot\left[\frac{\left(\varphi_s^{(K')^*}(\vec{x}_i)\beta\varphi(-\vec{k},s)\right)\left(\varphi^*(-\vec{k},s)C\varphi_t^{(K)^*}(\vec{x}'_j)\right)}{E_M - E_L - H_t - K_k} - \frac{\left(\varphi_t^{(K)^*}(\vec{x}_i)\beta\varphi(-\vec{k},s)\right)\left(\varphi^*(-\vec{k},s)C\varphi_s^{(K')^*}(\vec{x}'_j)\right)}{E_M - E_L - H_s - K_k}\right]\Bigg\}$$

(V-12)

In expression (V-12), $K_k = \hbar\, k\, c$ is the energy of the neutrino emitted in the first transition, $\varphi(k,s)$, for $s = 1,2$, are the two positive energy solutions of the one-particle Dirac equation:

$$k_\mu \gamma^\mu_{\alpha\beta}\, \varphi_\beta(\vec{k},s) = 0 \qquad \text{(V-13)}$$

which appear in the expansion of the neutrino field in plane waves:

$$\mathcal{U}(\vec{x}) = \mathcal{U}_+(\vec{x}) + C\,\overline{\mathcal{U}_+}(\vec{x}), \qquad \text{(V-2a)}$$

$$\mathcal{U}_{+,\alpha}(\vec{x}) = \frac{1}{(2\pi)^3}\sum_{s=1}^{2}\int d_3 k\, e^{-\vec{k}\cdot\vec{x}}\,\varphi_\alpha(\vec{k},s)\,\psi(\vec{k},s) \qquad \text{(V-14)}$$

where $\psi(k,s)$ satisfies the usual anticommutation relations:

$$\left\{ \psi(k,s)\ ,\ \psi(k',s') \right\} = 0 \qquad \text{(V-15a)}$$

$$\left\{ \psi(k,s)\ ,\ \psi^{\dagger}(k',s') \right\} = \delta_{ss'}\ \delta(k-k) \qquad \text{(V-15b)}$$

as necessary in order that (V-5) and (V-6) be satisfied.

Ψ_{M}, Ψ_{L} and Ψ_{N} in (V-12) are the nuclear wave functions of, respectively, the initial, intermediate and one of the final states[45], X representing all the nuclear coordinates and x_i that of the i^{th} nucleon. The operator Q_i is such that when applied to Ψ changes the i^{th} particle into a proton if it is a neutron and gives a zero result otherwise ($Q_i^2 = 0$). $\varphi_s^{(K)}(x_i)$ is a Coulomb wave function of an electron in the field of the charge Ze of the final nucleus, corresponding to the positive energy H_s. The upper index (K) corresponds to the several possibilities for the quantum numbers j, l and m. These wave functions should be properly normalized, as will be discussed later.

The four different types of terms in the expression (V-12) have the following origins.

α) The first two correspond to the emission of a neutrino with momentum $\hbar \vec{k}$ by the decaying i^{th} neutron, together with an electron of energy H_s (or H_t), and the subsequent reabsorption of the neutrino by the j^{th} decaying neutron, together with the emission of an electron of energy H_t (or H_s).

β) The last two terms correspond to the emission of a neutrino with momentum $-\hbar \vec{k}$ by the j^{th} neutron together with the emission of the electron with energy H_t (or H_s) and subsequent reabsorption of the neutrino by the i^{th} decaying neutron, with emission of the electron of energy H_s

45). In general there may be several states of the final atom with energy smaller than that of the initial one.

(or H_t). These two terms which were omitted by Furry are equal to the first two as can be seen by changing i into j and $\vec{k}$ into $-\vec{k}$; the third and fourth terms become equal, respectively, to the second and first. This adds a factor 2 to Furry's formula, corresponding to our V-12).

It is convenient to refer here to the Coulomb wave functions $\varphi^{(K)}$ for the electrons which occur in (V-12). As both electrons have small energies we restrict as usual the summation over K, K' in (V-12), to those for which the wave function is significant at $r = \rho$, the nuclear radius. These are the four wave functions $\varphi_{j,l,m}$ with $j = 1/2$, two of which correspond to $\ell = 0$ and $m = \pm 1/2$ and the other two to $\ell = 1$, $m = \pm 1/2$. In Dirac's representation of the matrices $\vec{\alpha}$, β they are[46],

$$\varphi^{(1)} = \varphi_{1/2,0,-1/2} = K(r,H) \begin{pmatrix} A_0 \sin\theta \, e^{-i\phi} \\ A_0 \cos\theta \\ 0 \\ B_0 \end{pmatrix} \tag{V-16a}$$

$$\varphi^{(2)} = \varphi_{1/2,0,1/2} = K(r,H) \begin{pmatrix} -A_0 \cos\theta \\ -A_0 \sin\theta \, e^{i\phi} \\ B_0 \\ 0 \end{pmatrix} \tag{V-16b}$$

$$\varphi^{(3)} = \varphi_{1/2,1,-1/2} = K(r,H) \begin{pmatrix} 0 \\ B_1 \\ A_1 \sin\theta \, e^{-i\phi} \\ A_1 \cos\theta \end{pmatrix} \tag{V-16c}$$

$$\varphi^{(4)} = \varphi_{1/2,1,1/2} = K(r,H) \begin{pmatrix} B_1 \\ 0 \\ -A_1 \cos\theta \\ -A_1 \sin\theta \, e^{i\phi} \end{pmatrix} \tag{V-16d}$$

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46) H.R.Hulme, Proc.Roy.Soc. 133A,312,1935.
M.E. Rose, Phys.Rev. 51, 484, 1937.

where:

$$A_\ell = (\gamma + i\lambda)\, e^{i\eta_\ell} - c.c. \qquad (H/mc^2 \mp 1)^{1/2} \qquad \text{(V-17a)}$$

$$B_\ell = (\gamma + i\lambda)\, e^{i\eta_\ell} + c.c. \qquad (H/mc^2 \pm 1)^{1/2} \qquad \text{(V-17b)}$$

where the upper signs correspond to $\ell = 0$ and the lower signs to $\ell = 1$ and

$$\gamma = (1 - \alpha^2 Z^2)^{1/2} \; ; \; \lambda = Z\,H/pc \; ; \; \alpha = e^2/\hbar c \qquad \text{(V-18)}$$

$$e^{2i\eta_\ell} = \frac{1 \pm i\alpha Z\, mc/p}{\gamma + i\lambda} \;, \; p = \sqrt{H^2 - m^2c^4}\;. \qquad \text{(V-19)}$$

Again the upper sign corresponds to $\ell = 0$ and the lower to $\ell = 1$. Finally

$$K(r,H) \cong \frac{1}{\pi\sqrt{8}} \left(\frac{mc}{\hbar}\right)^{3/2} \left(\frac{2rmc}{\hbar}\right)^{\gamma-1} \left(\frac{p}{mc}\right)^{\gamma-1/2} \frac{e^{\frac{\pi}{2}\alpha Z\frac{H}{pc}} \left|\Gamma\left(\gamma + i\alpha Z\,\frac{H}{pc}\right)\right|}{\Gamma(2\gamma + 1)} \qquad \text{(V-20)}$$

In the expression of $K(r,H)$ the following approximations were made

$$e^{ipr/\hbar} \approx 1$$

$$F(a,b;\, 2ipr/\hbar) \approx 1\;,$$

$F(a,b;x)$ being the hypergeometric confluent series. The normalization of these wave functions is such that the number of states in an interval of energy dW is given by $\frac{dW}{mc^2}$.

We now make the further usual approximation of taking the value of these wave functions at $r = \rho$ and also of neglecting the terms A_ℓ in relation to those in B_ℓ. This is a good approximation for medium heavy nuclei, say for $\alpha^2 Z^2 \approx 1/10$.

As a result of these simplifications the four wave functions $^{(K)}$ which will be used in (V-12) become functions only of ρ and H. Thus the expression (V-12) can be written as:

$$a_{N \leftarrow M}^{K,K'}(Hs) = \frac{g^2}{(2\pi)^3} \sum_L \sum_{i,j} \int d_3k \int \psi_N^* \beta_j Q_j^\dagger e^{i\vec{k}\cdot\vec{x}_j} \psi_L dX \int \psi_L^* \beta_i Q_i^\dagger e^{-i\vec{k}\cdot\vec{x}_j} \psi_M dX \cdot \left[\frac{\varphi_t^{(K)*}(\rho)\ (1+\vec{\alpha}\cdot\vec{n}) C \varphi_s^{(K')*}(\rho)}{E_M - E_L - H_s - \hbar kc} - \frac{\varphi_s^{(K')*}(\rho)\ (1+\vec{\alpha}\cdot\vec{n})\ C\varphi_t^{(K)*}(\rho)}{E_M - E_L - H_t - \hbar k c} \right] \tag{V-21}$$

with

$$n = \frac{\vec{k}}{k} ,$$

where we have used the property:

$$\sum_{s=1}^{2} \varphi_\alpha(\vec{k},s)\ \varphi_\beta^*(\vec{k},s) = \frac{1}{2}\left(1 + \vec{\alpha}\cdot\vec{n}\right)_{\alpha\beta} \tag{V-22}$$

Now the numerators of the two terms in (V-21) are equal since the operator

$$\beta (1 + \vec{\alpha}\cdot\vec{n})\ C$$

is symmetrical:

$$(\beta(1 + \vec{\alpha}\cdot\vec{n})\ C)^T = \beta(1 + \vec{\alpha}\cdot\vec{n})\ C. \tag{V-23}$$

Thus the expression in the square bracket in (V-21) can be written as:

$$\varphi_t^{(K)*}(\rho)(1 + \vec{\alpha}\cdot\vec{n})\beta\, C\, \varphi_s^{(K')*}(\rho) \cdot \frac{H_s - H_t}{(E_M - E_L - \hbar k c)^2} , \tag{V-24}$$

where H_s and H_t were neglected in relation to $\hbar k c$ as most of the contribution to the integral over k will come from large k's.

Now, in order to be able to make the summation over the intermediate states L in (V-21) by closure, we make a further approximation which is most unsatisfactory of those made in the present work. This will be to neglect $E_M - E_L$ in relation to $\hbar k c$ in the expression (V-24) for the bracket in (V-21).

$$\hbar k c - (E_M - E_L) \simeq \hbar k c \qquad \text{(V-25)}$$

This approximation can be somewhat justified in the following way. If we would use a Fermi gas model for the nucleus then only the single decaying neutron involved would suffer a recoil in the emission of the neutrino. Its change of energy should be, however, much smaller than $\hbar k c$ in view of its large mass. If the nucleon is tightly bounded to the other nucleons, however, as it actually is, then part of its recoil will be transmitted to the other nucleons. The effective mass of the nucleon is therefore increased, since there will be a tendency for it to carry along its neighboring particles, and therefore the energy given to the nucleus may be expected to be reduced even further than for the case of a Fermi gas model (the transmission of the recoil of the decaying neutron to the other particles is described formally by the use of appropriate nuclear wave functions). We should keep this approximation in mind when two terms in (V-21) cancel each other as a consequence of it. More accurately, the cancellation is not perfect. In this case (scalar theory) the terms in question are those coming from the $\vec{\kappa}\cdot\vec{n}$ term of the neutrino projection operator, when we consider the contributions from $Q_i^\dagger\ Q_j^\dagger$ and $Q_i^\dagger\ Q_i^\dagger$.

Thus making the approximation (V-25) and summing over the intermediate states by closure we obtain for (V-21) the expression:

$$a^{KK'}_{N\to M}(Hs) = 2\,\frac{g^2}{\hbar^2 c^2}(H_s - H_t)(\varphi_t^{(K)*}\beta\, C\, \varphi_s^{(K')*})\,\frac{1}{(2\pi)^3}\int\frac{d_3k}{k^2}\sum_{j<i}\int \psi_N^* Q_i^+ Q_j^+ \beta_i \beta_j e^{i\vec{k}\cdot(\vec{x}_j - \vec{x}_i)}\psi_M\, dX$$

(V-26

In the last step the terms in $Q_i^\dagger Q_j^\dagger$ and in $Q_j^\dagger Q_i^\dagger$ which appeared with the exponential factor $e^{i\vec{k}\cdot(\vec{x}_i-\vec{x}_j)}$ and $e^{-i\vec{k}(\vec{x}_i-\vec{x}_j)}$, respectively, were added up, after chaning $\vec{k}$ into $-\vec{k}$ for the second one. In doing this the terms (i,j) and (j,i) which included the factor 1 from the neutrino projection operator became equal[47] and those which included the factor $\vec{\alpha}\cdot\vec{n}$ cancelled each other. Now, as was pointed out previously in connection with the approximation of neglecting $E_K - E_L$ in presence of $\hbar k c$, this cancellation is not complete. However, the correct result would surely be much smaller than the one we would get by keeping only the term (i,j) and forgetting (j,i). This last result, it is easy to see, would be of the same order of magnitude as (V-26). However, the selection rules of the nuclear matrix element would be in this case those of a polar vector, say $\Delta i = 0, \pm 1$ but no $0 \longrightarrow 0$, with change of parity. As the transition between the ground states will be of the type $0 \longrightarrow 0$ this term will give no contribution to the transition probability to the ground state. Even if there were a slightly excited state of the final nucleus with spin 1 and appropriate parity the contribution of this term will be small in relation to that of (V-26). We think, thus that we are justified in neglecting this term.

If we now carry out the integration over k in (V-26) we obtain:

$$\sum_{K,K'} \left| a^{KK'}_{N\leftarrow M}(Hs) \right|^2 = \frac{g^4}{h^2 c^4}(H_s - H_t)^2 \sum_{K,K'=1}^{4} \left| \varphi_t^{(K)*}(\rho)\; C\varphi_s^{(K')*}(\rho) \right|^2 \left| \iint \psi_N^* \frac{\beta_1\beta_2}{r_{12}}\, \psi_M\, dX \right|^2 \qquad \text{(V-27)}$$

where $r_{12} = x_1 - x_2$

In (V-27) it was assumed that the final nucleus differs from the initial one by the decay of the pair (1,2) of neutrons into protons.

In order to evaluate $\sum_{K,K'} \left| \varphi_t^* \beta C \varphi_s^* \right|^2$ in a compact way

47). This brings about another factor 2, besides the one referred to before, which was omitted in Furry's computation.

it is convenient to introduce the matrix formed with the wave functions $\varphi_\alpha^{(K)}$ (K, = 1,2,3,4):

$$M(\rho,H) = \begin{pmatrix} \varphi_1^{(1)} & \varphi_1^{(2)} & \varphi_1^{(3)} & \varphi_1^{(4)} \\ \varphi_2^{(1)} & \varphi_2^{(2)} & \varphi_2^{(3)} & \varphi_2^{(4)} \\ \varphi_3^{(1)} & \varphi_3^{(2)} & \varphi_3^{(3)} & {}_3^{(4)} \\ \varphi_4^{(1)} & \varphi_4^{(2)} & \varphi_4^{(3)} & \varphi_4^{(4)} \end{pmatrix} =$$

$$= K(\;\;,H) \begin{pmatrix} 0 & 0 & 0 & B_1(H) \\ 0 & 0 & B_1(H) & 0 \\ 0 & B_o(H) & 0 & 0 \\ B_o(H) & 0 & 0 & 0 \end{pmatrix} \qquad (V\text{-}28)$$

(compare (V-16)). In (V-28) the angular depending terms in $\varphi^{(K)}$ were neglected, as already justified.

Now we define a new matrix N:

$$N = M^\dagger M . \qquad (V\text{-}29).$$

It is clear from (V-28) that, as the matrix element $M_{\lambda\mu}$ of M is equal to $\varphi_\mu^{(\lambda)}$ (with λ, μ = 1,2,3,4), the matrix elements of N are :

$$N_{\mu\nu} = \sum_{\lambda=1}^{4} M^*_{\lambda\mu} M_{\lambda\nu} = \sum_{\lambda=1}^{4} \varphi_\mu^{(\lambda)*} \varphi_\nu^{(\lambda)} .$$

Thus we see that N is a kind of projection operator. Its use will simplify the computations, specially in the other types of theory (vector, for instance).

We find for N:

$$N(\rho,H) = 2K^2(\rho,H) . \begin{pmatrix} |B_o|^2 & & & \\ & |B_o|^2 & & \\ & & |B_1|^2 & \\ & & & |B_1|^2 \end{pmatrix} = N^T(\rho,H) \qquad (V\text{-}30)$$

(which is true in Dirac's representation).

We can now express N as:

$$N(\rho,H) = \frac{1}{2} K^2(\rho,H) \left[(1+\beta) |B_0|^2 + (1-\beta) |B_1|^2 \right] =$$

$$= \frac{1}{2} K^2(\rho,H) \left(a H + b mc^2 \beta \right) \qquad \text{(V-31)}$$

where:

$$a = 2(1+\gamma) \simeq 4 \; ; \quad b = 2(1+\gamma-\alpha^2 Z^2) \simeq 4 \qquad \text{(V-32)}$$

or:

$$N(\rho,H) = 2 K^2(\rho,H) \left(H + mc^2 \beta \right) . \qquad \text{(V-33)}$$

Using the matrix N we can perform the summation over K and K' in (V-27):

$$\sum_{K,K'=1}^{4} \left| \varphi_t^{(K)*}(\rho) \beta \, c \, \varphi_s^{(K')}(\rho) \right|^2 = \mathrm{spur}\left[\beta C N(\rho,H_s) C \beta N(\rho,H_t) \right]$$

$$= 16 K^2(\rho,H_s) K^2(\rho,H_t) (H_s H_t - m^2 c^4) \qquad \text{(V-34)}$$

Now taking (V-20), (V-27) and (V-34) into (V-11) we find for the total probability of double β decay per unit time:

$$P_1 = \frac{18}{25\pi^3} \left[\frac{1}{2} \Gamma(2\gamma+1) \right]^{-4} \left(\frac{2\rho mc}{\hbar} \right)^{4(\gamma-1)} \cdot \left(\frac{g}{\hbar c} \right)^4 \left(\frac{mc}{\hbar} \right)^6 \frac{mc^2}{\hbar} \frac{\alpha^2 Z^2}{\rho^2} \varphi(\varepsilon-2) \cdot$$

$$\cdot \left| \{ \beta_1 \beta_2 \}_{MN} \right|^2 \qquad \text{(V-35)}$$

where:

$$\varphi(\varepsilon-2) = \int_1^{\varepsilon-1} h_s(\varepsilon-h_s)(\varepsilon-2h_s)^2 \left[h_s(\varepsilon-h_s) - 1 \right] dh_s \qquad \text{(V-36)}$$

In (V-36) the quantities ε and h_s are given by:

$$h_s = \frac{H_s}{mc^2} \; , \quad \varepsilon = \frac{E_M - E_N}{mc^2} \qquad \text{(V-37)}$$

In (V-35) the approximation was made[43]:

$$e^{\pi\alpha Z\frac{H}{pc}} \left| \Gamma(\gamma - i\alpha Z \frac{H}{pc}) \right|^2 \approx 2\pi\alpha Z \frac{H}{pc} , \qquad (V-38)$$

which is very good for $H \approx mc^2$ and inaccurate only by 20 % for $H \gg mc^2$, if $\alpha Z \approx 0.3$. Also the following approximation was made:

$$\left\{ \frac{\beta_1\beta_2}{r_{12}} \right\}_{MN} = \int \Psi_N^* \frac{\beta_1\beta_2}{r_{12}} \Psi_M \, dx \approx \frac{6}{5\rho} \{\beta_1\beta_2\}_{MN} , \quad (V-39)$$

by substituting $\frac{1}{r_{12}}$ in the integrand by its mean value over the nuclear volume. It should be mentioned here that, had we substituted, before making the integration over k, $e^{i\vec{k}\cdot\vec{r}_1}$ by its average over the nuclear volume, as done by Furry, we would have obtained the same result (V-38).

The function φ defined by (V-36) is the following:[44]

$$\varphi(x) = \frac{x^4}{3} \left(1 + \frac{11}{10}x + \frac{x^2}{5} + \frac{x^3}{70} \right) . \qquad (V-40)$$

Now, in order to have the expression (V-35) finally in an appropriate form for the numerical evaluation of the half life for the double β-decay, which is given by:

$$t = \frac{0.69}{P} \qquad (V-41)$$

it is convenient to consider the order of magnitude of g and of the nuclear matrix elements.

First, as the nucleons are heavy particles and the transitions involve non-relativistic energies, we should make the usual simplifications[48],

48). H. A. Bethe and R. F. Bacher, Rev.Mod.Phys.8,88,1936.

$$\beta \to 1 \;;\; \gamma_5 \to (\overline{\tfrac{v}{c}}) \cdot 1 \;,\; \beta\vec{\sigma} \to \vec{\sigma} \;,\; \vec{\alpha} \to \overline{\tfrac{v}{c}}\,\vec{\sigma} \;,\; \beta\vec{\alpha} \to \overline{\tfrac{v}{c}}\,\vec{\sigma} \qquad \text{(V-41)}$$

where[49],

$$(\overline{\tfrac{v}{c}})^2 \approx \frac{1}{1000} \qquad \text{(V-41a)}$$

Now, the value of g, which can be determined from the β -decay of light nuclei by taking $\left\{ 1 \right\} \approx 1$ is:

$$g \cong 4 \times 10^{-49} \text{ erg cm}^3 \qquad \text{(V-42)}$$

This value corresponds to a half life of about 15 minutes for the neutron[50]

For the medium heavy elements:

$$g_{eff} = g \cdot \left\{ 1 \right\} \approx 4 \times 10^{-50} \text{ erg cm}^3 \;, \qquad \text{(V-42)}$$

determined in Fermi's original paper, or:

$$\left| \left\{ 1 \right\} \right|^2 \approx 1/100 \qquad \text{(V-43)}$$

The quantity g_{eff}, whose value is given by (V-42) is frequently called the coupling constant for heavy nuclei (in which case the matrix element $\left\{ 1 \right\}$ is taken as equal to 1).

---

49). The average value of $\frac{v}{c}$ appearing in (V-41) is frequently taken as $\frac{1}{10}$ (compare ref. 46). A more appropriate evaluation is as follows. The square of the matrix γ^5 corresponds to the quantity $(\frac{v}{c})^2$ whose average value in the nucleus is about 1/30. The square of the matrix γ^5 is, however, of the order of $\frac{1}{3000}$ because all matrix elements which are non diagonal can be expected to be smaller than the corresponding averages (E.P.Wigner, private communication). Thus for heavy nuclei we take $\left|\left\{\gamma^5\right\}\right|^2 \approx \frac{1}{1000}\left|\left\{1\right\}\right|^2$.

50). A. Snell, Science 108, 167, 1948.

In our case the matrix element $\{1\}_{MN}$ is between two states differing by the transformation of two neutrons, instead of one, into protons. However, expression (V-43) should still give a reasonable order of magnitude of the square of the nuclear matrix element for medium heavy nuclei. Thus we shall use for g the value 4×10^{-49} erg cm^3 and for the square of the matrix elements (the factors $\frac{\vec{v}}{c}$ excluded) the value 1/100.

It will be seen later that using these numerical values in expression (V-35), whose selection rules are $\Delta i = 0$ with no change of parity (appropriate for a $0 \longrightarrow 0$ (no) transition), will lead to a half life for $_{50}Sn^{124}$ of the order of the one found experimentally by Fireman. This is to be contrasted with Furry's results[44] which would lead to a half life 10^3 times larger than the experimental value.

The origin of this discrepancy is the following. First, a factor 4 is missing in Furry's matrix elements (and then a factor 16 in the transition probability) because some terms were neglected by him, as already referred to. Second, in view of his one intermediate state approximation he obtains in his equation corresponding to (V-36):

$$\left| \{\beta_1\}_{ML} \{\beta_2\}_{LN} \right|^2 \approx 10^{-4}$$

instead of the term:

$$\left| \{\beta_1 \beta_2\}_{MN} \right|^2 \approx 10^{-2}$$

b) Pseudoscalar theory.

A pure pseudoscalar theory is known to be unsatisfactory for single β-decay. However, we refer here to the results of this theory for the sake of completeness.

In this case the transition probability will be given by (V-35) with $\{\beta_1 \beta_2\}_{MN}$ changed into $\{\beta_1 \gamma_1^5 \beta_2 \gamma_2^5\}_{MN}$ (which has the same selection rules as $\{\beta_1 \beta_2\}$).

Now for a pure pseudoscalar theory a value of the coupling constant g_{PS} larger than that for the scalar theory (g_S) is needed in order that the theoretical half life for single β decay be of the same order of the experimental one. In other terms in comparing the results of the scalar and pseudoscalar theories we should take:

$$\left| g_{PS} \{\beta \gamma^5\} \right| \approx \left| g_S \{\beta\} \right| \qquad \text{(V-44)}$$

In consequence of (V-44) the value of g_{PS}^2 will be 10^3 times [49] larger than that of g_S^2. However, the γ_5's appearing in the matrix elements of the pseudoscalar theory nearly compensate this larger value of g. Thus, in opposition to the case of single β decay, the analysis of the double β-decay in the pseudoscalar theory does not lead to an argument against this theory.

c) Pseudovector theory.

This is considered the best simple theory for the description of single β decay.

It is convenient here to introduce a factor $\frac{1}{\sqrt{3}}$ in the expression of the pseudovector interaction:

$$\mathcal{H}_{PV} = \frac{1}{\sqrt{3}} \; g \, \overline{\Psi}_P \sigma^{\mu} \Psi_N \; \overline{\varphi}_e \sigma_{\mu} \, \mathcal{U} + \text{h.c.} \qquad \text{(V-45)}$$

where

$$\sigma^{\mu} = (\gamma^5, \vec{\sigma}),$$

in order that the value of g computed from the β decay of the neutron and of the light nuclei be the same as the one used in the scalar theory (compare V-42).

Now, proceeding similarly to the scalar case, we obtain an expression for $a^{KK'}_{N \leftarrow M}$ similar to (V-21), where the quantity:

$$\beta_j \beta_i \varphi_t^* \beta (1 + \vec{\alpha}\cdot\vec{n}) C \varphi_s^*$$

is substituted by

$$\frac{1}{3} \sigma_j^{\mu} \sigma_i^{\nu} \varphi_t^* \sigma_\mu (1 + \vec{\alpha}\cdot\vec{n}) \sigma_\nu C \beta_T \varphi_s^* = T \qquad (V- 46)$$

Here:

$$\sigma_\mu \sigma_i^\nu = - \gamma^5 \gamma_i^5 + \vec{\sigma}\cdot\vec{\sigma}_i$$

Now, if we expand the expression (V-46), we find:

$$T = S + A \qquad (V-47)$$

where S is symmetrical in the exchange of the states s,t and A is anti-symmetrical:

$$S = \frac{1}{3} \varphi_s^* \Big[\gamma_i^5 \gamma_j^5 + \vec{\sigma}_i \cdot \vec{\sigma}_j + \gamma_i^5 \gamma_j^5 (\vec{\alpha}\cdot\vec{n}) - (\gamma_i^5 \vec{\sigma}_j + \vec{\sigma}_i \gamma_j^5)\cdot(\vec{n} + \vec{\alpha}) +$$

$$+ \Big\{ (\vec{\sigma}_i \cdot \vec{\sigma}_j) \vec{n} + \vec{\sigma}_i \wedge (\vec{n} \wedge \vec{\sigma}_j) + \vec{\sigma}_j \wedge (\vec{n} \wedge \vec{\sigma}_i) \Big\} \cdot \vec{\alpha} \Big] C \beta_T \varphi_t^* \simeq$$

$$\simeq \frac{1}{3} \varphi_s^* \Big[\vec{\sigma}_i \cdot \vec{\sigma}_j + \Big\{ (\vec{\sigma}_i \cdot \vec{\sigma}_j) \vec{n} + \vec{\sigma}_i \wedge (\vec{n} \wedge \vec{\sigma}_j) + \vec{\sigma}_j \wedge (\vec{n} \wedge \vec{\sigma}_i) \Big\} \cdot \vec{\alpha} \Big] C \beta_T \varphi_t^*$$

$$(V-48)$$

$$A = \frac{1}{3} \varphi_s^* \Big[(\vec{\sigma}_i \wedge \vec{\sigma}_j) \cdot \vec{\sigma} - (\vec{\sigma}_i \wedge \vec{\sigma}_j) \cdot \vec{n} \gamma^5 + (\gamma_i^5 \vec{\sigma}_j - \vec{\sigma}_i \gamma_j^5) \wedge \vec{n} \cdot \vec{\sigma} \Big] C \beta_T \varphi_t^*$$

$$(V-49)$$

In (V-48) terms were neglected which, in view of the presence of γ_i^5 or γ_j^5, were small in relation to others in the same expression with the <u>same selection rules</u>, and same parity in relation to the exchange of i and j and change of $\vec{n}$ into $-\vec{n}$. For instance $\gamma_i^5 \gamma_j^5$ and $(\gamma_i^5 \vec{\sigma}_j + \vec{\sigma}_i \gamma_j^5)\cdot\vec{n}$ (scalars) were neglected in relation to $\vec{\sigma}_i \cdot \vec{\sigma}_j$. On the other hand, we did not neglect $(\gamma_i^5 \vec{\sigma}_j - \gamma_j^5 \vec{\sigma}_i)\wedge\vec{n}\cdot\vec{\sigma} \approx$ $\approx (\frac{\vec{v}}{c})(\vec{\sigma}_j - \vec{\sigma}_i)\wedge\vec{n}\cdot\vec{\sigma}$ in relation to $\vec{\sigma}_i \wedge \vec{\sigma}_j \cdot \vec{\sigma}$, which has the same selection rule because this last term, changing sign when $i \rightarrow j$ $\vec{n} \longrightarrow -\vec{n}$, will be cancelled in the following computation.

It should be observed that S is also symmetric in the indices i,j and A is antisymmetric in i,j.

Now, in the same way as in the scalar theory (for which only symmetrical terms existed), when we add the two terms in (V-21) corresponding to the exchange of s and t, we will get a factor

$$\frac{H_s - H_t}{(E_M - E_L - \hbar k c)^2}$$

as in (V-24), for the symmetric terms.

For the antisymmetric terms, however, we get a factor:

$$\frac{2}{E_M - E_L - \hbar k c} \qquad \text{(V-50)}$$

Thus, as most of the contributions to the integral over k will come from large k's, the antisymmetrical terms will be <u>formally</u> larger than the symmetrical ones. Actually their contributions to the total probability will be, formally, about 10^3 times the contribution of the symmetrical terms.

However, they may vanish in virtue of the selection rules, in which case the main terms will be the symmetric ones. This is actually the case when the only state of the final atom with an energy smaller than that of the initial atom is the ground state. In this case (as both the initial and final nuclei have zero spin and same parity) the terms with a selection rule different of $\Delta i = 0$ (no) will vanish.

If we now make the approximation

$$\left| E_L - E_M - \hbar k c \right| \longrightarrow \hbar k c \qquad \text{(V-25)}$$

and sum by closure, changing $\vec{k}$ into $-\vec{k}$ for the terms j,i, as before, we find that from the symmetrical terms (which are also symmetrical in i,j) only the terms not containing $\vec{n}$ (only $\vec{\sigma}_i \cdot \vec{\sigma}_j$, therefore) will survive. From the antisymmetrical terms (also antisymmetrical in i,j) only the terms containing $\vec{n}$ will survive $(\vec{\sigma}_i \wedge \vec{\sigma}_j \cdot \vec{n})\gamma^5$ and $(\vec{\sigma}_i \gamma_j^5 - \vec{\sigma}_j \gamma_i^5) \wedge \vec{n} \cdot \vec{\sigma}$ in the case.

Here again, we should remember that this cancellation of terms (i,j) with (j,i) should only be partial as we made the approximation (V-25). However, we should not worry about this as the cancelled term $(\vec{\sigma}_i \wedge \vec{\sigma}_j \cdot \vec{\sigma})$ has the selection rule: $\Delta i = \pm 1, 0$ but not $0 \longrightarrow 0$, (no). This is the same as for the term $(\gamma_i^5 \vec{\sigma}_j - \vec{\sigma}_i \gamma_j^5) \wedge \vec{n} \cdot \vec{\sigma}$, which was retained. We do not think that the cancellation of the term $(\vec{\sigma}_i, \vec{\sigma}_j, \vec{\sigma})$ was so imperfect that it could give a contribution to P larger than that of the retained term with the same selection rule.

Proceeding now with the computation in the same way as for the scalar theory we find for the transition probability P for double β decay:

$$P = P_1 + P_2 + P_2' \qquad \text{(V-51)}$$

In (V-51) P_1 is given by (V-35) with $\{\beta_1 \beta_2\}_{MN}$ substituted by $\frac{1}{3}\{\vec{\sigma}_1 \cdot \vec{\sigma}_2\}_{MN}$ (both of which have the same selection rule, say $\Delta i = 0$ (no)).

P_2, P_2' are given by:

$$P_2 = f \sum_N \chi(\varepsilon_N - 2) \left| \left\{ (\gamma_1^5 \vec{\sigma}_2 - \vec{\sigma}_1 \gamma_2^5) \wedge \frac{(\vec{x}_1 - \vec{x}_2)}{|\vec{x}_1 - \vec{x}_2|} \right\}_{MN} \right|^2 \qquad \text{(V-52a)}$$

$$P_2' = f \sum_N \chi'(\varepsilon_N - 2) \left| \left\{ \sigma_1 \wedge \sigma_2 \cdot \frac{\vec{x}_1 - \vec{x}_2}{\vec{x}_1 - \vec{x}_2} \right\}_{MN} \right|^2 \qquad \text{(V-52b)}$$

where:

$$f = \frac{9}{8\pi^3} \left[\frac{1}{2} \Gamma(2\gamma + 1) \right]^{-4} \left(\frac{2\rho mc}{h}\right)^{4(\gamma - 1)} \left(\frac{\varepsilon}{\hbar c}\right)^4 \left(\frac{mc}{\hbar}\right)^4 \frac{mc^2}{\hbar} \frac{\alpha^2 \lambda^2}{4}$$

The selection rules for the terms P_2 and P_2' are:

α) For P_2:

$\Delta i = \pm 1$ or 0, but not $0 \to 0$ (no)

β) For P_2':

$\Delta i = 0$ (yes)

In (V-52) a factor $\dfrac{1}{|\vec{x}_1 - \vec{x}_2|^2}$ was substituted by its mean value over the nuclear volume: $\dfrac{9}{4\rho^2}$. The functions χ[44] and χ' are:

$$\chi(\varepsilon - 2) = \int_1^{\varepsilon - 1} h_s(\varepsilon - h_s) \left[h_s(\varepsilon - h_s) - 1 \right] d\, h_s \qquad \text{(V-53)}$$

$$\chi'(\varepsilon - 2) = \int_1^{\varepsilon - 1} h_s(\varepsilon - h_s) \left[h_s(\varepsilon - h_s) + 1 \right] d\, h_s \qquad \text{(V-54)}$$

or:

$$\chi(x) = x^2 \left(1 + \frac{7}{6} x + \frac{x^2}{3} + \frac{x^3}{30}\right) \qquad \text{(V-53a)}$$

$$\chi'(x) = 2x + 3x^2 + \frac{3}{2} x^3 + \frac{x^4}{3} + \frac{x^5}{30} \qquad \text{(V-54a)}$$

λ and λ' are of the same order of magnitude.

In order to compare the predictions of the pseudovector theory with that of the scalar theory we observe that:

α) For the transition to the ground state the two theories lead to the same result as then (transition $0 \longrightarrow 0$, (no)) only P_1 in (V-51) survives. Now the term P_1 in (V-51) differs from the corresponding transition probability in the scalar case only by the substitution:

$$\left|\{\beta_1\ \beta_2\}\right|^2 \longrightarrow \frac{1}{9}\left|\{\vec{\sigma}_1 \cdot \vec{\sigma}_2\}\right|^2$$

Now the shell model predicts a singlet-singlet transition for the involved nucleons; in this case we have:

$$\left|\{\vec{\sigma}_1 \cdot \vec{\sigma}_2\}\right| = 3 \cdot \left|\{1\}\right| \approx 3\left|\{\beta_1 \beta_2\}\right|$$

Thus the two theories lead to the same result for the transitions to the ground state ($0 \longrightarrow 0$, (no)).

β) If there was a slightly excited state of the final nucleus with spin zero and opposite parity to that of the ground state than the term P_2' in (V-51) would lead to a transition to such state as its selection rule is:

$$\Delta i = 0 \quad \text{(yes)}.$$

In this case, if we assume:

$$\left|\left\{\vec{\sigma}_1 \wedge \vec{\sigma}_2 \cdot \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|}\right\}\right|^2 \approx 10^{-2}$$

(compare with (V-43)) then we find, for a value $\xi \sim 5$, that P_2' is about 10^3 times larger than P_1. Thus most of the double β decay, in such case, would correspond to transitions to this excited state and the half life would be of about 10^{12} years. However, we shall give in sec. C an argument against such a possibility.

γ) If there is a slightly excited state of the final nucleus with spin 1 and same parity as the ground state then P_2 will give a contribution to the transition probability. If we evaluate the matrix element in P_2 as:

$$\left| \left\{ \gamma_1^5 \vec{\sigma}_2 - \gamma_2^5 \vec{\sigma}_1 \right) \wedge \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|} \right\}_{MN} \right|^2 \approx \overline{\left(\frac{v}{c}\right)^2} \cdot 10^{-2} \approx 10^{-5}$$

we find that, for $\xi \sim 5$:

$$P_2 \approx P_1$$

δ) Vector theory:

Here, making the similar decomposition of the s,t matrix element, as before, into the symmetrical and antisymmetrical parts, S and A, we obtain:

$$S = \varphi_s^*(1 + \vec{\alpha} \cdot \vec{n}) \; c\beta_T \varphi_t^* \qquad \text{(V-55)}$$

$$A = i\, \varphi_s^* \left[(\vec{\alpha}_i - \vec{\alpha}_j) \wedge \vec{n} \cdot \vec{\tau} - (\vec{\alpha}_i \wedge \vec{\alpha}_j) \cdot \vec{n}\, \gamma^5 \right] \qquad \text{(V-56)}$$

where, as before, terms were neglected in relation to larger ones with the same selection rules and same parity in relation to the transformation:

$$i \longleftrightarrow j, \quad \vec{k} \longrightarrow -\vec{k}$$

Here the result differs from that of the pseudovector theory by

α) The matrix appearing in P_1 is $\{1\}_{MN}$

β) The matrix element appearing in P_2, given by (V-52a), is:

$$3\left\{ (\vec{\alpha}_1 - \vec{\alpha}_2) \wedge \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|} \right\}_{MN} \approx 3 \cdot \left(\frac{v}{c}\right) \cdot 10^{-1}$$

The factor 3 here appearing comes from the fact that in the pseudovector case we introduced a normalization factor $\frac{1}{\sqrt{3}}$ (compare (V-45)).

γ) In P_2' given by (V-52b):

$$3\left\{(\vec{\alpha}_1 \wedge \vec{\alpha}_2)\cdot \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|}\right\}_{MN} \approx 3 \cdot \overline{\left(\frac{v}{c}\right)^2} \cdot 10^{-1}$$

The order of magnitude of P_1 and P_2 is here the same as in the pseudovector case (P_2 is actually larger than in the pseudovector case by a factor 9), but P_2' will become smaller than P_1 in view of the factor $\left(\frac{v}{c}\right)^4$. One should also observe that the corresponding terms of the vector and pseudovector theories have the same selection rules.

δ) Tensor theory:

Here we find for S and A:

$$S = + \beta_i\beta_j\varphi_s^*\left[\vec{\sigma}_i\cdot\vec{\sigma}_j + \left\{(\vec{\sigma}_i\cdot\vec{\sigma}_j)\vec{n} + \vec{\sigma}_i\wedge(\vec{n}\wedge\vec{\sigma}_j) + \vec{\sigma}_j\wedge(\vec{n}\wedge\vec{\sigma}_i)\right\}\cdot\vec{\alpha}\right] C\beta_T\varphi_t^* \quad \text{(V-57)}$$

$$A = -\frac{1}{3}\beta_i\beta_j\varphi_s^*\left[\vec{\sigma}_i\wedge\vec{\sigma}_j\cdot\vec{\sigma} - \vec{\sigma}_i\wedge\vec{\sigma}_j\cdot\vec{n}\,\gamma^5 - (\beta_i\gamma_i^5 - \beta_j\gamma_j^5)(\vec{\sigma}_i\cdot\vec{\sigma}_j)(\vec{n}\cdot\vec{\sigma})\right]C\beta_T\varphi_t^* \quad \text{(V-58)}$$

[illegible] (V-57) and (V-58) with (V-48) and (V-49) we see that the tensor theory leads to the same result as the pseudovector theory.

f) Angular correlation.

For the analysis of the angular correlation between the two emitted electrons we use, as usual, plane wave functions for them.

In view of the previous analysis there are only three cases to be considered

α) If the selection rule is:

$\Delta i = 0$ (no)

β) If the selection rule is:

$\Delta i = 0$ (yes)

γ) If the selection rule is:

$$\Delta i = 0, -1 \text{ but not } 0 \longrightarrow 0 \quad (\text{no})$$

In case α), for which, as it was seen before, all theories give the same result $P_1(H_s)$ for the energy spectrum, this expression, before summing over the electron states, is proportional to:

$$\sum_{\sigma_s, \sigma_t = 1}^{2} \left| \varphi^*(\vec{k}_s, \sigma_s) \, c \beta_T \, \varphi^*(\vec{k}_t, \sigma_t) \right|^2 \qquad (V\text{-}59)$$

the summation being only over positive energy states.

Thus we find for (V-59), by the usual spur method:

$$1 + \frac{\vec{P}_s \cdot \vec{P}_t}{H_s \, H_t} c^2 - \frac{m^2 c^4}{H_s H_t} \qquad (V\text{-}60)$$

Thus the angular distribution in all the theories, for the terms of type α) is the one given by (V-60), thus favouring the emission of the two electrons in the same direction. This type of angular distribution is the one that should be expected for transitions of the type:

$$0 \longrightarrow 0 \quad (\text{no}) \qquad (V\text{-}61)$$

which is the one predicted by the shell model if the only accessible final state is the ground state.

In case β), which occur only for pseudovector and tensor theories, we have instead of (V-59)

$$\sum_{\sigma_s, \sigma_t = 1}^{2} \left| \varphi^*(\vec{k}_s, \sigma_s) \, \gamma^5 \, c \beta_T \, \varphi^*(\vec{k}_t, \sigma_t) \right|^2$$

and instead of (V-60) we find an angular correlation of the type:

$$1 + \frac{\vec{P}_s \cdot \vec{P}_t}{H_s \, H_t} c^2 + \frac{m^2 c^4}{H_s \, H_t} \qquad (V\text{-}62)$$

which still favour the emission of the two electrons in the same direction, although not so strongly as in the first case in view of the positive sign of the mass term.

Nevertheless we will give in part C an argument in favour of the vanishing of P_2^1, by the selection rules for the actual possibilities of double β decay.

Finally in case γ), which corresponds to the term P_2 appearing only in the vector, tensor and pseudovector theories, we find an angular distribution of the type:

$$1 - \frac{\vec{p}_s \cdot \vec{p}_t}{H_s\, H_t}\, c^2 - \frac{m^2 c^4}{H_s H_t} \qquad \text{(V-63)}$$

Thus in this case the emission of the two electrons in opposite directions is favoured.

2) Mixed theories.

As referred in Part IV the only mixed theories which lead to double β decay with no neutrinos are those of the types (a,b), (a,c), (b,d) and c,d). The last two reduce to the first ones if the neutrino mass vanishes. We consider, then, only those of the first two types. Also, in these cases, we restrict ourselves to those theories obtained from a <u>usual</u> simple Fermi interaction by one of the substitutions:

a) $\psi_\nu \longrightarrow \psi_\nu + \lambda \gamma_5 C \overline{\psi_\nu}$ (Toushek Theories[38])

b) $\psi_\nu \longrightarrow \psi_\nu + \lambda\, C\, \overline{\psi_\nu}$ (Racah Theories[20])

The predictions of these theories for double β decay without neutrinos are as follows:

a) Toushek theories (type (a,c))

No double β decay without neutrinos occurs in such theories[38] for any value of the constant λ (see Part IV).

For $\lambda = 1$ these theories reduce to Fireman theories. Now as we found in the analysis of invariance under time inversion only the Fermi interaction of the type considered by Fireman would be invariant if the neutrino field transforms under time inversion according to the types of transformation III or IV (which are possible only for fields of zero mass). The experimental results of Fireman which indicate the existence of double β decay with no neutrinos lead us to the exclusion of these two types of transformation under time inversion (III and IV) for the neutrino field.

b) Racah theories (type (a,b)).

If we proceed to the computation of the probability for double β-decay in a mixed theory of the type (a,b) (say for the scalar one with the interaction given by (V-8)), taking into account both the cases when the intermediate neutrino is a particle or an antiparticle, we find the same result as for the corresponding Majorana theory, but for a factor:

$$\frac{4|\lambda|^2}{(1+|\lambda|^2)^2} \qquad \text{(V-64)}$$

(see formula (V-8)), which is in general smaller than the unity, except in the case when $\lambda = 1$ (Furry projection theory) when it becomes equal to 1. (Thus we see, once again, that Furry projection theory leads to the same result as Majorana theory).

3) Summary.

We have found in the previous sections the following results on the double β decay:

a) A theory of the type (a,c) in which the neutrino field ψ_ν

(Dirac field) appears in the combination $\psi_\nu + \lambda\gamma_5 C\,\overline{\psi}$, say in the scalar case:

$$\mathcal{H} = g\,\overline{\psi}_P\psi_N\overline{\psi}_e(\psi_\nu + \lambda\gamma_5 C\overline{\psi}_\nu) + \text{h.c.},$$

do not lead to double β decay without neutrinos[38].

b) A theory of the type (a,b) in which ψ_ν (Dirac field) appears in the combination $\psi_\nu + \lambda\,C\,\overline{\psi_\nu}$, say in the scalar case:

$$\mathcal{H} = \frac{g}{\sqrt{1+|\lambda|^2}}\,\overline{\psi}_P\,\psi_N\,\overline{\psi}_e\,(\psi_\nu + \lambda\,C\,\overline{\psi}_\nu) + \text{h.c.}$$

leads to double β decay without neutrinos. The probability for this process is related to the value predicted by the corresponding Majorana theory (substitute $\psi_\nu + \lambda\,C\,\overline{\psi}_\nu$ by the Majorana field $\mathcal{U}$) by the reduction factor:

$$\frac{4|\lambda|^2}{(1+|\lambda|^2)^2}$$

If $\lambda = 1$ (Furry projection theory) such a theory (in which the neutrino is a Dirac particle) gives the same result as the corresponding Majorana theory.

c) Majorana theories.

Here we consider the five simple types of interaction:

Scalar: $g\,\overline{\psi}_P\psi_N\,\overline{\psi}_e\,\mathcal{U} + \text{h.c.}$

Vector: $g\,\overline{\psi}_P\gamma^\mu\psi_N\overline{\psi}_e\,\gamma_\mu\,\mathcal{U} + \text{h.c.}$

Tensor: $\frac{g}{\sqrt{3}}\,\overline{\psi}_P\,\gamma^{\mu\nu}\psi_N\,\overline{\psi}_e\,\gamma_{\mu\nu}\mathcal{U} + \text{h.c.}$ $\left(\gamma^{\mu\nu} = \frac{1}{2i}[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu]\right)$

Pseudovector: $\frac{g}{\sqrt{3}}\,\overline{\psi}_P\gamma^5\gamma^\mu\psi_N\,\overline{\psi}_e\,\gamma^5\gamma_\mu\mathcal{U} + \text{h.c.}$

Pseudoscalar: $g_{PS}\,\overline{\psi}_P\gamma^5\psi_N\,\overline{\psi}_e\,\mathcal{U} + \text{h.c.}$

where:

$$g \approx 4 \cdot 10^{-49} \quad \text{erg cm}^3$$

as obtained from the β decay of the light nuclei and of the neutron (notice the factor $\frac{1}{\sqrt{3}}$ in the pseudovector and tensor cases). In the pure pseudoscalar theory:

$$g^2_{PS} \approx 10^3 \; g^2$$

We find that

α) For the transitions to the ground state of the final nucleus all theories predict the same result for the probability of double decay:

$$P_1 = K \, \varphi(\varepsilon - 2) \, |\eta_1|^2$$

where

$$K = \frac{18 \times 16}{75\pi^3 \left|\Gamma(2\gamma+1)\right|^4} \left(\frac{g}{\hbar c}\right)^4 \left(\frac{mc}{\hbar}\right)^6 \frac{mc^2}{\hbar} \frac{\alpha^2 Z^2}{\rho^2} \left(\frac{2\rho\, mc}{\hbar}\right)^{4(\gamma-1)}$$

(compare (V-35). η_1 is given, in the several theories, by:

Scalar: $\{\beta_1 \beta_2\}_{MN} \approx 10^{-1}$

Vector: $\{1\}_{MN} \approx 10^{-1}$

Tensory: $\frac{1}{3}\{\beta_1 \vec{\sigma}_1 \cdot \beta_2 \vec{\sigma}_2\}_{MN} \approx 10^{-1}$

Pseudovector: $\frac{1}{3}\{\vec{\sigma}_1 \cdot \vec{\sigma}_2\} \approx 10^{-1}$

Pseudoscalar: $(10)^3 \cdot \{\gamma^5_1 \; \gamma^5_2\}_{MN} \gtrsim 10^{-1}$

The order of magnitude of P_1 is the same for all cases:

$$P_1 \approx \frac{1}{2} \cdot 10^{-15} \text{ years}^{-1} \qquad \text{for } \varepsilon \sim 5$$

The selection rule is also the same:

$\Delta i = 0$ (no)

The shape of the energy spectrum is given by:

$$h(\mathcal{E}-h)\ (\mathcal{E}-2h)\ \left[2h\ (\mathcal{E}-h)-1\ \right]$$

where h is the energy of one of the electrons in unities m c^2. The spectrum is symmetrical in relation to h = $\frac{\mathcal{E}}{2}$ and vanishes for this value of h.

Finally the angular correlation of the emitted electrons is also the same for all theories:

$$\alpha = 1 + \frac{\vec{p}_s \cdot \vec{p}_t}{W_s W_t} c^2 - \frac{m^2 c^4}{W_s W_t}$$

Thus the emission of the two electrons in the same direction is favored.

β) Only for the vector, pseudovector and tensor theories the transition to a final state with spin 1 and opposite parity to that of the ground state are allowed. The probability of such transitions is given by

$$P_2 = \frac{25}{18} K\ \chi(\mathcal{E}_N - 2) \left|\eta_2\right|^2$$

(compare (V-52a)). $\left|\eta_2\right|^2$ is given by:

Vector: $9 \left| \left\{ \vec{\alpha}_1 - \vec{\alpha}_2 \wedge \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|} \right\} \right|^2 \approx 9 \cdot 10^{-5}$

Tensor: $\left| \left\{ (\beta_1 \gamma_2^5 - \beta_2 \gamma_1^5)\ (\vec{\sigma}_1 \cdot \vec{\sigma}_2)\ \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|} \right\} \right|^2 \approx 10^{-5}$

Pseudovector: $\left| \left\{ (\gamma_1^5 \vec{\sigma}_2 - \vec{\sigma}_1 \gamma_2^5) \wedge \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|} \right\} \right|^2 \approx 10^{-5}$

The order of magnitude of P_2 is the same in the tensor and pseudovector theories:

$$P_2 \approx P_1 \approx 10^{-15} \text{ years}^{-1} \quad (\text{for } \mathcal{E} \sim 5)$$

and nine times larger in the vector theory.

The selection rule for P_2 in all three theories is:

$$\Delta i = \pm 1, 0, \text{ but not } 0 \longrightarrow 0 \quad \text{(no)}.$$

The shape of the energy spectrum is given by:

$$h(\mathcal{E}-h)\left[h(\mathcal{E}-h)-1\right],$$

the same in all three cases. We see that it is symmetrical in relation to $h = \frac{\mathcal{E}}{2}$ and have a maximum at this point.

Also the angular correlation is the same in these three theories:

$$\alpha = 1 - \frac{\vec{p}_s \cdot \vec{p}_t}{H_s H_t} c^2 - \frac{m^2 c^4}{H_s H_t}$$

Thus, in opposition to the term P_1, this term leads to preferencial emission of the two electrons in opposite directions.

γ) The tensor and pseudovector interactions lead also to transitions with the selection rule:

$$\Delta i = 0 \quad \text{(yes)},$$

which should not be expected, however, to occur in the actual cases of double β decay, as it will be verified in the following section.

The probability of such transitions is given by:

$$P_2' = \frac{25}{16} K \chi' (\mathcal{E}_n - 2) \left| \eta_2' \right|^2$$

(compare (V-52b). $\left|\eta_2'\right|^2$ is given by:

$$\text{Tensor: } \left| \left\{ \beta_1 \vec{\sigma}_1 \wedge \beta_2 \vec{\sigma}_2 \cdot \frac{\vec{x}_1 - \vec{x}_2}{\left|\vec{x}_1 - \vec{x}_2\right|} \right\} \right|^2 \approx 10^{-2}$$

$$\text{Pseudovector: } \left| \left\{ \vec{\sigma}_1 \wedge \vec{\sigma}_2 \cdot \frac{\vec{x}_1 - \vec{x}_2}{\left|\vec{x}_1 - \vec{x}_2\right|} \right\} \right|^2 \approx 10^{-2}$$

The order of magnitude of P_2' (if such a term were allowed in actual cases) is:

$$P_2' \approx 10^{-12} \text{ years}^{-1} \qquad (\text{for } \mathcal{E} \sim 5)$$

The shape of the spectrum is given by

$$k(\varepsilon - k)[k(\varepsilon - k) + 1].$$

It is practically the same as for the term P_2.

The angular correlation is here:

$$\alpha = 1 + \frac{\vec{p}_s \cdot \vec{p}_t}{H_s H_t} c^2 + \frac{m^2 c^4}{H_s H_t}$$

It favours the emission of the two electrons in the same direction, but not as strongly as in P_1.

C. Analysis of the experimental results.

The experimental data on double β decay is still very scarce and not completely satisfactory. The only determination of the half life for a double β decay which seems conclusive and does not relay on semiempirical evaluation of the available energy is Fireman's[51] one, for the decay of ${}_{50}Sn^{124}$ into ${}_{52}Te^{124}$.

This brilliant experiment definitely eliminates the possibility of the neutrino being a Dirac particle with the usual type of interaction, as the value obtained for the half life, of the order of 10^{16} years, is too small to agree with the value 10^{24} years [51] obtained from formula (V-9). Even if we would take the unreasonable maximum value 1 for the nuclear matrix element the theoretical result, 10^{20} years, would be, still, much larger than the experimental one.

However, if we compute the half-life for double β decay in Majorana theory, for a $0 \longrightarrow 0$ (no) transition, (as both the initial and the final nucleus are of the even-even type), we find a half life (using formula (V-35)):

$$t_1 = 3 \times 10^{15} \text{ years} \qquad \text{(V-65)}$$

51). E. Fireman, Phys. Rev. 74, 1238, 1948; 75, 323, 1949.

for $Z = 52$, $\rho = 8 \times 10^{-13}$ cm, $\mathcal{E} = 5$ (or for 1.65 MeV kinetical energy of the emitted electrons). This result, as well as the energy spectrum and angular correlation (given in (V-60) and (V-56), respectively), which is the same for all types of simple interaction, is in good agreement with Fireman's experimental value[51].

On the other hand, if $_{52}Te^{124}$ had also zero spin but apposite parity to that of $_{52}Te^{124}$ (which is practically excluded) the pseudovector and tensor theories would predict a half-life:

$$t_{\frac{1}{2}} = 2 \times 10^{12} \text{ years} \qquad \text{(V-66)}$$

much smaller than the observed one ($\sim 5 \cdot 10^{15}$ y). Although this possibility is excluded by theoretical reasons (is impossible in the shell model) one might worry about the possibility that one of the low excited states of the final nucleus would have zero spin and opposite parity to that of the ground state. This is not the case for $_{52}Te^{124}$ as, besides other reasons, no isomeric state of this nucleus has been observed in this region. In general, no isomeric state has been observed for even-even nuclei (Mattauch's rule [52]), with exception of $_{82}Pb^{204}$ and $_{32}Ge^{72}$, the first case not very certain and the second very probably not of opposite parity to that of the ground state, although its spin may be also 0[53].

This is an unhappy situation from the experimental point of view as it excludes the possibility of having in some cases quite small life times for double β-decay via the term (V-53) for the pseudovector and tensor theories (see (V-65)).

However, there are good reasons to expect, especially in Fireman's

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52). J. Mattauch, Zeits.f.Physik, 117, 246, 1941.

53). E.Segrè and A.C. Helmholz, Rev.Mod.Phys. 21, 271,1949.

case that a spectrum of the type $P_2(\Pi_s)$ (see integrand of (V-53)) but of the order of magnitude of P_1 would be superimposed to the spectrum $P_1(\Pi_s)$ (which corresponds to the decay to the ground state) if some of the low excited states of $_{52}Te^{124}$ have spin 1 and same parity as the ground state[53]. This would happen for the terms:

1) $\vec{\sigma}_i \wedge \vec{\sigma}_j \cdot \vec{\sigma}$, for the Pseudovector and Tensor theories

(see (V-49) and (V-58)) whose total cancellation in the previous computation is impaired by the approximation (V-50).

2) P_2 (given by (V-62) for the pseudovector, tensor and vector theories (with the corresponding matrix elements given in sec. A). In the case of the double β decay of Sn^{124}, if the first excited level of Te^{124} (0.6 MeV above the ground state) has spin 1 and same parity as the ground state (as is probably the case[53]) then we would have:

$$\frac{1}{0.69} \; P_2 \; \simeq \; \frac{1}{5} \cdot 10^{-15} \;\; \text{years}^{-1} \qquad \text{(V-67a)}$$

in the pseudovector and tensor theories and a value nine time larger in the vector theory. This fact that the vector theory would lead, in such case, to a half life 9 times smaller than the experimental one should not be taken, however, as an argument against this theory because our evaluation of the matrix element is surely a rough approximation.

Finally some of the forbidden terms which were not considered might produce a similar effect.

These arguments point out to the necessity of the experimental determination of the energy spectrum for the double β -decay of $_{52}Sn^{124}$, as well as to the need of a theoretical analysis of the forbidden spectra in the several theories.

Although, as was pointed out before, no discrimination among the several theories is possible if there is only one transition to the ground state (of the type $0 \longrightarrow 0$ (no)), as will be probably the case for ${}_{34}Se^{80} \longrightarrow {}_{36}Kr^{80}$ [53], it is possible that this analysis for the case of ${}_{50}Sn^{124}$ would lead to a further information about the type of interaction involved.

In what refers to the mixed theories of the type (a,b), (or better of Racah's type), the half life for which is increased by the factor $\frac{(1+|\lambda|^2)^2}{4|\lambda|^2}$ (see (V-64) that we should have $|\lambda| \gtrsim \frac{1}{8}$ as the value $\frac{1}{8}$ leads already to an increase by a factor 16 of the half life (V-65).

Finally, in what concerns the more recent investigations on other cases of double β -decay[54] is should be observed that the experimental half lifes are, in all these cases, much smaller than those predicted by M. Goeppert Mayer's formula (V-9) for the usual type of theories. This should be considered as a new argument against these theories. On the other side the fact that these half lifes are also much larger than the value predicted by (V-35) with $g_{eff} = g|\{\ \}|_{MN} \approx 4 \times 10^{-50}$ erg cm^3 should not be considered as an argument against the Majorana theory as we know that also for simple β-decay, in many cases when the spectrum is clearly of the allowed shape, the value of g_{eff} is much smaller than 4×10^{-50} erg cm^3, which simply indicates that the nuclear wave function of the initial and final states are strongly dissimilar. Also these experiments are not as conclusive as Fireman's one.

54). M.G. [illegible] and I.H. Reynolds, Phys.Rev.76, 1265,1949; 76,[illegible]

C. A. Levine, A. Ghiorso and G.T.Seaborg, Phys.Rev.77,296,1950.

D. Conclusions:

From the preceding analysis we conclude that Majorana theory, with any kind of simple interaction , leads to a result for the double β decay in reasonable agreement with experiment, in contrast to the usual theories with conservation of particles, which predict too large a value for the lifetime. This does not necessarily mean that the neutrino is a Majorana particle (or, as have been sometimes improperly said, that the anti-neutrino is equal to the neutrino), because a Dirac type of neutrino in a mixed theory of the type (a,b) would lead to the same result, for a not too small $|\lambda|$.

This analysis points to the necessity of making a more systematic study of all possible cases of double β-decay, because the magnitude of $g_{eff} = g\,|\langle\ \rangle_{MN}|$ may be in some cases larger than $4 \cdot 10^{-50}$ erg cm^3 (as happens in many cases of simple β-decay). If g_{eff} is this large, we might obtain an experimentally significant gain in the probability of the process.

Also the analysis of the energy spectrum should be carried out, when possible, as it might give more information on the nature of the interaction. This analysis would, probably, bring a new strong argument against the usual type of theory for which the energy spectrum (given by (V-10)) is completely different from that predicted by Majorana theory. The verification of the constancy of the sum of the energies of the two electrons would also exclude definitively the usual type of theories. Finally, it would be interesting to check the angular correlation given by (V-60), (V-63), as any tendency to isotropy will indicate the presence of a vector, tensor or pseudovector interaction. It should be observed that, in principle, the experimental analysis of the angular correlation in double β decay is simpler than in single β decay as, here, the two emitted electrons may be, both, detected. Also, as P_1 corresponds to transitions to the ground state and P_2 to an excited state, these terms can be, in principle, separated as the sum of the energies of the two electrons is different for these two terms.

On the theoretical side, the forbidden transitions should be analysed especially for $_{52}Sn^{124}$, as some of the antisymmetric terms may give a contribution of the order of the allowed (symmetric) transition to the ground state.

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LIST OF REFERENCES†

1). S. Tomonaga, Prog. Theor. Phys. 1, 27, 1946.

2). J. Schwinger, Phys. Rev. 74, 1439, 1948; 75, 651, 1949; 76, 790, 1949.

3). R.P. Feynman, Phys.Rev. 76, 749, 769, 1949.

4). E.C.G.Stueckelberg and D.Rivier, Helv.Phys.Acta, 23, 215, 1949.

5). W.Heisenberg, Zeits,f.Phys. 120, 513, 673, 1943; Zeits,f.Naturf. 1, 608, 1946. See also J.A. Wheeler, Phys. Rev. 52, 1107, 1937.

6). F.J.Dyson, Phys. Rev. 75, 486, 1736, 1949.

7). P.A.M. Dirac, Phys.Rev.73, 1092, 1948.

8). H. Yukawa, Phys. Rev. 77, 219, 849, 1950

9). A.Pais and G. Uhlenbeck, Phys. Rev. 79, 145, 1950.

10). E.L. Fireman, "An experiment on double β -decay", Princeton thesis, 1948.

12). E.P. Wigner, Gottinger Nachrichten, p. 546, 1932.

13). T.D.Newton and E. P. Wigner, Rev. Mod. Phys. 21, 400, 1949.

14). A.S.Wightman and E. P. Wigner, forthcoming paper on Relativistic Invariance of Quantum Theory

15). W.Pauli, Hanbuch der Physik, Vol. 24I,p.259; Rev. Mod.Phys. 13, 203, 1941.

16). K. M. Case, Phys. Rev. 76, 1, 1949.

20). G. Racah, Nuovo Cim. 14, 322, 1937.

21). W.Pauli, Ann. Inst. H.Poincaré, 6, 109, 1936.

22). C.N.Yang and J.Tiomno, Phys. Rev. 79, 495, 1950.

26). E.Majorana, Nuovo Cim. 14, 171, 1937.

27). W.H. Furry, Phys.Rev. 54, 56, 1938.

29). E.Schrödinger, Berl.Ber. p. 418, 1930.

31). V.Fock, Zeits,f.Phys. 75, 622, 1932.

32). R.Becker and G. Liebfried, Phys.Rev.69, 341, 1946.

---

†). The numbers of the references correspond to the order in which they appear in the text as footnotes.

33). G.Wentzel, Quantum Theory of Fields, 1949, Interscience Publishers, New York.

35). J. Tiomno, Phys. Rev. 76, 856, 1949.

36). E.J. Konopinski and G.E.Uhlenbeck, Phys. Rev. 48, 7, 1935.

37). E. Fermi, Zeits.f.Phys. 88, 161, 1934.

38). B. Toushek, Zeits.f. Phys. 125, 108, 1948.

39). M.Fierz, Zeits.f.Phys. 104, 553, 1937.

40). C.L.Critchfield and E. P. Wigner, Phys. Rev. 60, 412, 1941; C.L. Critchfield, Phys. Rev. 63, 417, 1943.

41). L.Michel, Proc. London Phys. Soc., 63A, 514, 1950.

42). R.B.Leighton, C.D. Anderson and A.J. Seriff, Phys. Rev. 75, 1432, 1949.

43). M.Goeppert Mayer, Phys. Rev. 48, 512, 1935.

44). W.H. Furry, Phys. Rev. 56, 1184, 1939.

46). H.R. Hulme, Proc.Roy. Soc. 133A, 380, 1931
M.E. Rose, Phys. Rev. 51, 484, 1937.

48). H.A. Bethe and R. F. Bacher, Rev. Mod. Phys. 8, 82, 1936

50). A. H. Snell, Science 108, 167, 1948; A. H. Snell, F. Pleasonton and R. V. McCord, Phys.Rev. 78, 310, 1950. See also: J. M. Robson, Phys. Rev. 78, 311, 1950.

51). E.Fireman, Phys. Rev. 74, 1238, 1948; 75, 323, 1949.

52). J. Mattauch, Zeits.f.Phys. 117, 246, 1941.

53). E.Segrè and A.C. Helmholz, Rev. Mod. Phys. 21, 271, 1949.

54). M.G. Inghram and J. H. Reynolds, Phys. Rev. 76, 1265, 1949; 78, 822, 1950.
C.A. Levine, A. Ghiorso and G.T. Seaborg, Phys. Rev. 77, 296, 1950.

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